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SOME RAY TRACING PROBLEMS RELATED TO CIRCLES AND ELLIPSES

Henning Bach

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A classical ray tracing problem, namely the celebrated Alhazen Mirror Problem, is reviewed. An alternative formulation of the law of reflection, leading, in the spherical mirror case, directly to Alhazen's equation is presented. The locus method of ray tracing is introduced and applied to Alhazen's problem. Finally the technique is applied to some ray tracing problems, which often occur in practice, namely reflection by a circular cylinder and diffraction by a circular edge. Although these problems cannot in general be solved analytically, a technique is devised that may save considerable amounts of computer time in actual applications.

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1. INTRODUCTION

Practical applications of optical and quasi-optical methods always imply solution of ray tracing problems. Such problems are usually very easy to formulate, but analytical solutions are in general not available. They are therefore normally solved numerically and in practical applications of the geometrical optics and diffraction theories the major part of the computer time is spent solving ray tracing problems.

We shall consider a number of ray tracing problems associated with circles and ellipses. Such problems arise in applications involving reflection by spheres and circular and elliptic cylinders and diffraction by circular and elliptic edges. We shall assume that these bodies are conductors imbedded in a homogeneous dielectric. Thus the rays we are to consider are segments of straight lines and refracted rays are excluded. For the sake of convenience we shall assume throughout the report that the refractive index is unity.

Our point of departure will be a classical ray tracing problem, namely that of reflection by a spherical mirror. Although not well known, this problem was originally formulated in an ingenious way that allowed an analytic solution. This present study leads to an alternative formulation of the law of reflection, which directly produces the simple equation for the determination of the reflection point. Furthermore, an alternative ray tracing technique, the locus method, is introduced and applied to the ray tracing problems considered.

(Received for publication 5 April 1989)

2. ALHAZEN'S MIRROR PROBLEM

2.1 The Original Formulation

This famous problem stems from the Arabic mathematician Alhazen (987–1038). In his *Optics* the problem has the following form: Find the points on a concave spherical mirror at which a ray of light coming from a given point must strike in order to be reflected to another given point. The beginnings of this problem are found in Ptolemy's works on optics, and after Alhazen's masterly, however complicated, discussion of it, it became famous in Europe on account of the geometrical difficulties to which it gave rise [1].

In the following we shall review Alhazen's problem in the form given by Dorrie [2].In Figure 1, $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ are the two given points while $R(x, y)$ is a reflection point on the circle with radius r. For later use we also introduce polar coordinates for these points:

$$\begin{aligned} r_1 &= \sqrt{x_1^2 + y_1^2} \qquad & \tan\alpha_1 &= y_1/x_1 \\ r_2 &= \sqrt{x_2^2 + y_2^2} \qquad & \tan\alpha_2 &= y_2/x_2 \\ r &= \sqrt{x^2 + y^2} \qquad & \tan\alpha &= y/x. \end{aligned} \tag{1}$$

We denote the angle of incidence by v_1 and the angle of reflection by v_2 and furthermore introduce the angles u_1 and u_2 between the x-axis and the rays RP_1 and RP_2, respectively. We then have

$$v_1 = \alpha - u_1$$

$$v_2 = u_2 - \alpha$$

and from the law of reflection $v_1 = v_2$. Introducing

$$\tan\alpha = \frac{y}{x}; \qquad \tan u_1 = \frac{y - y_1}{x - x_1}; \qquad \tan u_2 = \frac{y - y_2}{x - x_2} \tag{2}$$

into the equation

$$\tan v_1 = \tan v_2. \tag{3}$$

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[1]Cajori, F. (1962) A History of Physics. Dover publications, New York, pp. 21-23.

[2]Dorrier, H. (1965) A Hundred Great Problems of Elementary Mathematics. Dover Publications, New York, pp. 197-200.

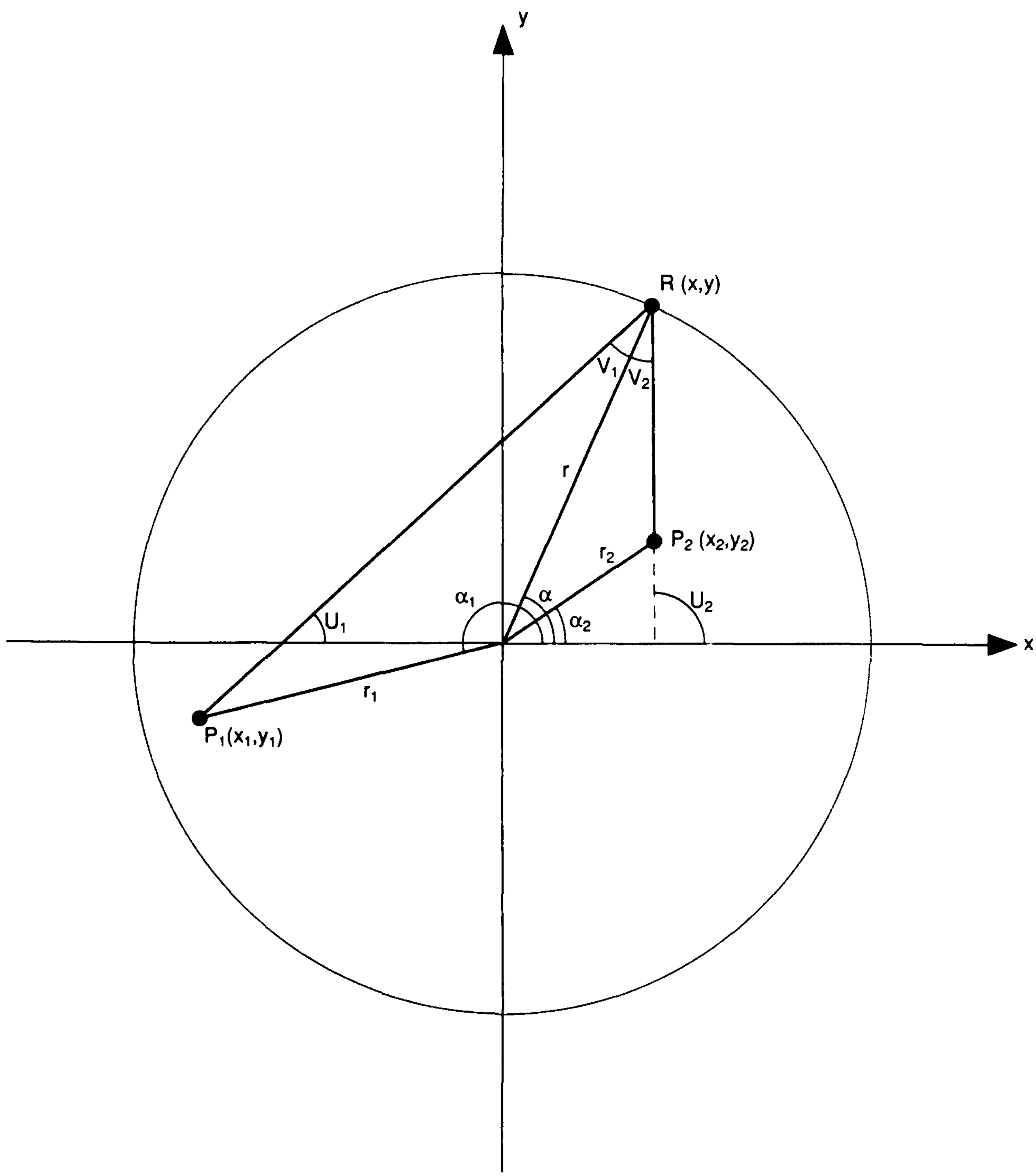


Figure 1. Alhazen's Mirror Problem.

we obtain after some reductions

$$\frac{xy_1 - yx_1}{x^2 + y^2 - xx_1 - yy_1} = -\frac{xy_2 - yx_2}{x^2 + y^2 - xx_2 - yy_2}. \tag{4}$$

Now, since we are interested only in the reflection points on the circle, we may substitute r^2 for $x^2 + y^2$, so that the formula can be written

$$\frac{xy_1 - yx_1}{r^2 - xx_1 - yy_1} = -\frac{xy_2 - yx_2}{r^2 - xx_2 - yy_2}. \tag{5}$$

Reduction to a standard form yields

$$Ax^2 - Ay^2 + 2Cxy + r^2(2Dx + 2Ey) = 0 \tag{6}$$

where

$$\begin{aligned} A &= x_1y_2 + x_2y_1 = r_1r_2\sin(\alpha_1 + \alpha_2); \qquad D = -0.5(y_1 + y_2) \\ C &= y_1y_2 - x_1x_2 = -r_1r_2\cos(\alpha_1 + \alpha_2); \qquad E = 0.5(x_1 + x_2). \end{aligned} \tag{7}$$

Since the curves given by Eq. (6) are conic sections and since these have in general four points of intersection with a circle, it may be concluded that there are at most four reflection points on a spherical mirror.

We neglect the case where both points are at the center of the circle, such that there is an infinity of reflection points.

This result was obtained by Alhazen. The determination of the reflection points involves solving a quartic equation, the theory of which was not known to Alhazen. Thus, as we shall see, the problem had to wait 800 years for its solution.

2.2 Discussion of Alhazen's Curve

The conic section represented by Eq. (6) is a simple hyperbola with asymptotes that are perpendicular to each other. Since no constant term is present, the curve contains the origin. From the eigenvalue equation

$$\begin{vmatrix} A - \lambda & C \\ C & -A - \lambda \end{vmatrix} = 0$$

we obtain the eigenvalues

$$\lambda = \pm\sqrt{A^2 + C^2} = \pm r_1 r_2 \tag{8}$$

where we have used Eqs. (7) and (1). To determine the center (x_c, y_c) of the hyperbola we solve the linear equations

$$Ax_c + Cy_c + Dr^2 = 0$$

$$Cx_c - Ay_c + Er^2 = 0$$

and find, again using Eq. (7)

$$(x_c, y_c) = \frac{r^2}{2}\left(\frac{x_1}{r_1^2} + \frac{x_2}{r_2^2}, \quad \frac{y_1}{r_1^2} + \frac{y_2}{r_2^2}\right). \tag{9}$$

The angle ω by which we rotate the coordinate system to remove the xy-term from Eq. (6) is found from

$$\begin{Bmatrix} A-\lambda & C \\ C & -A-\lambda \end{Bmatrix} \begin{Bmatrix} \cos\omega \\ \sin\omega \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

which yields for example

$$\tan\omega = \frac{C}{A+\lambda}.$$

Inserting $\lambda = r_1 r_2$ we obtain, after some reductions

$$\omega = 0.5(\alpha_1 + \alpha_2) - \frac{\pi}{4}. \tag{10}$$

The new coordinate system C_1 thus has its origin at the center C and axes which are at 45 degree angles with the angular bisector of the angle defined by the radii through P_1 and P_2.

It may be shown through a lengthy calculation that the equation of the hyperbola in C_1 is

$$x^2 - y^2 = 2k; \qquad k = x_c y_c \tag{11}$$

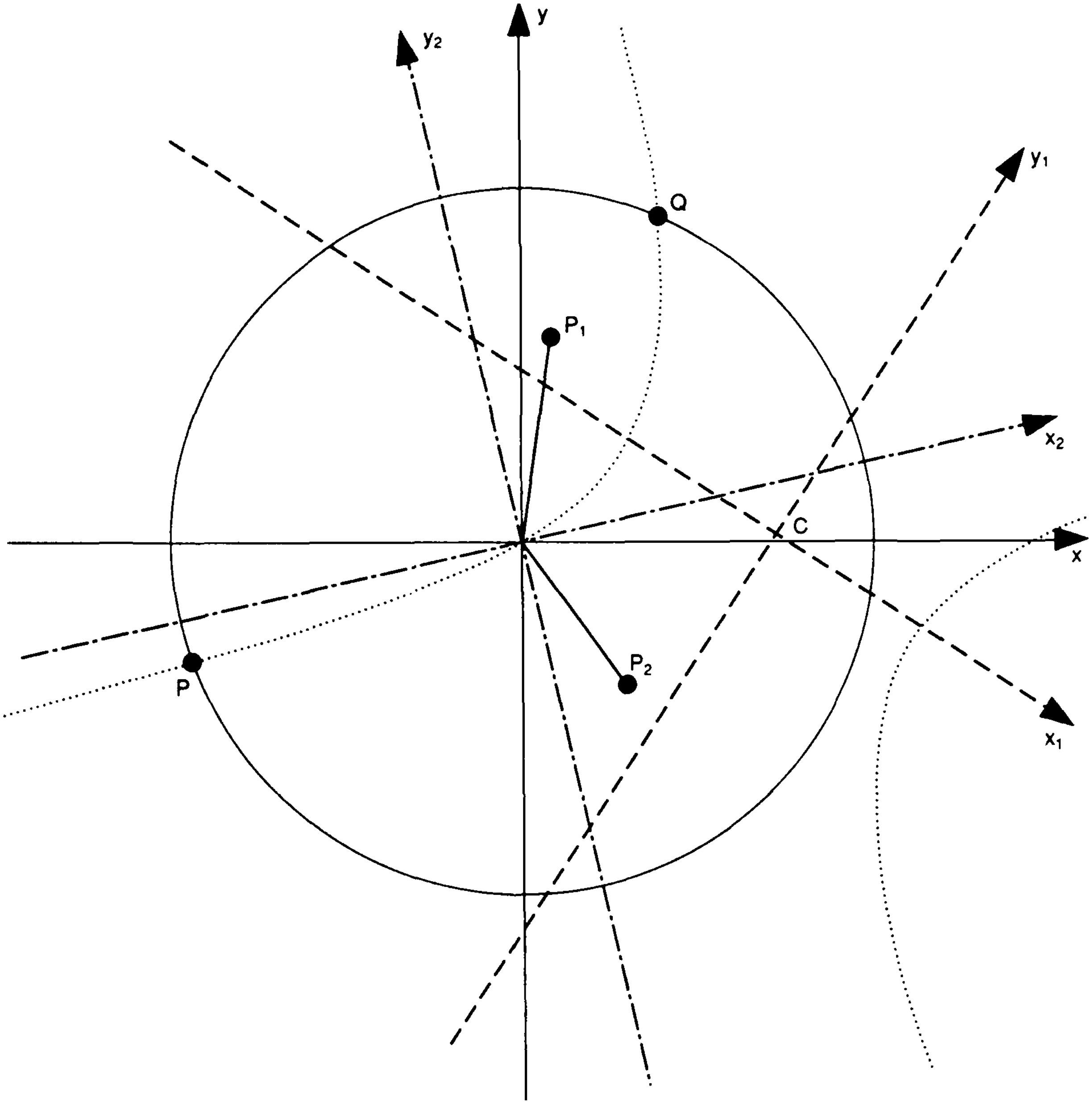
where (x_c, y_c) are the coordinates of the center C in the new coordinate system C_1, given by

$$(x_c, y_c)_1 = \frac{r^2}{r_1 r_2}\left(\frac{r_2 + r_1}{2}\cos\frac{\alpha_1 + \alpha_2}{2}, \qquad \frac{r_2 - r_1}{2}\sin\frac{\alpha_1 + \alpha_2}{2}\right) \tag{12}$$

which yields

$$k = \frac{r^4}{8} = \frac{r_2^2 - r_1^2}{r_1^2 r_2^2}\sin(\alpha_1 + \alpha_2). \tag{13}$$

Figure 2 shows an example of Alhazen's hyperbola for $r = 10$, $(x_1, y_1) = (1, 6)$ and $(x_2, y_2) = (3, -2)$. The axis marked x_2 is the angular bisector of the angle between the radii through P_1 and P_2. In this example only two points of reflection exist, namely P and Q, only one branch of the hyperbola intersects the circle.



REFLECTION BY CIRCLE: INPUT DATA

x_1 =	1.000	y_1 =	6.000	x_2 =	3.000	y_2 =	−4.000	R =	10.000
R_1 =	6.083	a_1 =	80.538	R_2 =	5.000	a_2 =	−53.130	v =	13.704

COEFFICIENTS OF POLYNOMIAL

a =	14.000	c =	−27.000	d =	−1.000	e =	2.000

CONSTANTS OF HYPERBOLA

xc =	7.351	yc =	0.108	xm =	2.000	ym =	1.000
xc_1 =	7.168	yc_1 =	−1.637	k =	−11.730		

ANALYTICAL REFLECTION POINTS AT FI = 67.16 199.16
NUMERICAL REFLECTION POINTS AT FI = 67.16 199.16

Figure 2. Alhazen's hyperbola (dotted) and related coordinate systems.

2.3 An Analytic Solution to the Mirror Problem

In 1861 the Danish mathematician Tychsen published a simple analytic solution to Alhazens problem [3]. Probably because his results were published in Danish, they never became internationally known. His solution is based on a theorem that he published in 1851 [4]. This theorem states:

When the real coefficients in the equation of the fourth degree

$$x^4 + ax^3 + bx - 1 = 0 \tag{14}$$

satisfy the condition

$$\left(\frac{a+b}{4}\right)^{\frac{2}{3}} - \left(\frac{a-b}{4}\right)^{\frac{2}{3}} + 1 \underset{>}{\overset{<}{=}} 0 \tag{15}$$

the equation has four different real roots, three real roots or two real roots, respectively.

In order to utilize this theorem, the equation of the hyperbola, as given by Eq. (11) is transformed to a coordinate system C_2 with its origin at the center of the circle and with axes parallel and perpendicular to the bisector mentioned earlier. The transformation equation is

$$\begin{Bmatrix} x \\ y \end{Bmatrix}_2 = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} \begin{Bmatrix} x \\ y \end{Bmatrix}_1 + \begin{Bmatrix} x_c \\ y_c \end{Bmatrix}_1$$

which yields

$$xy - y_c x - x_c y = 0 \tag{17}$$

in the coordinate system C_2, where (x_c, y_c) is given by Eq. (12). Next, to derive an equation of the form of Eq. (14), a new variable z is introduced, and we rewrite the equation of the circle in the form

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[3]Tychsen, C. (1859) On a Mathematical Investigation of a Billiard Problem. (In Danish). *Matematisk Tidsskrift*, Copenhagen, pp. 60–64.

[4]Tychsen, C. (1851) A Remark Concerning an Equation of the Fourth Degree. (In Danish). *Matematisk Tidsskrift*, Copenhagen, pp. 141–144.

$$z = \frac{r+x}{y} = \frac{y}{r-x}. \tag{18}$$

Solving these equations for x and y we find

$$x = r\frac{1-z^2}{1+z^2}; \qquad y = r\frac{2z}{1+z^2}. \tag{19}$$

These expressions, when inserted into Eq. (17), lead to the fourth degree algebraic equation

$$z^4 - 2\frac{r+x_c}{y_c}z^3 + 2\frac{r-x_c}{y_c}z - 1 = 0. \tag{20}$$

which is of the same form as Eq. (14). Insertion of the coefficients in the condition [Eq. (15)] then yields

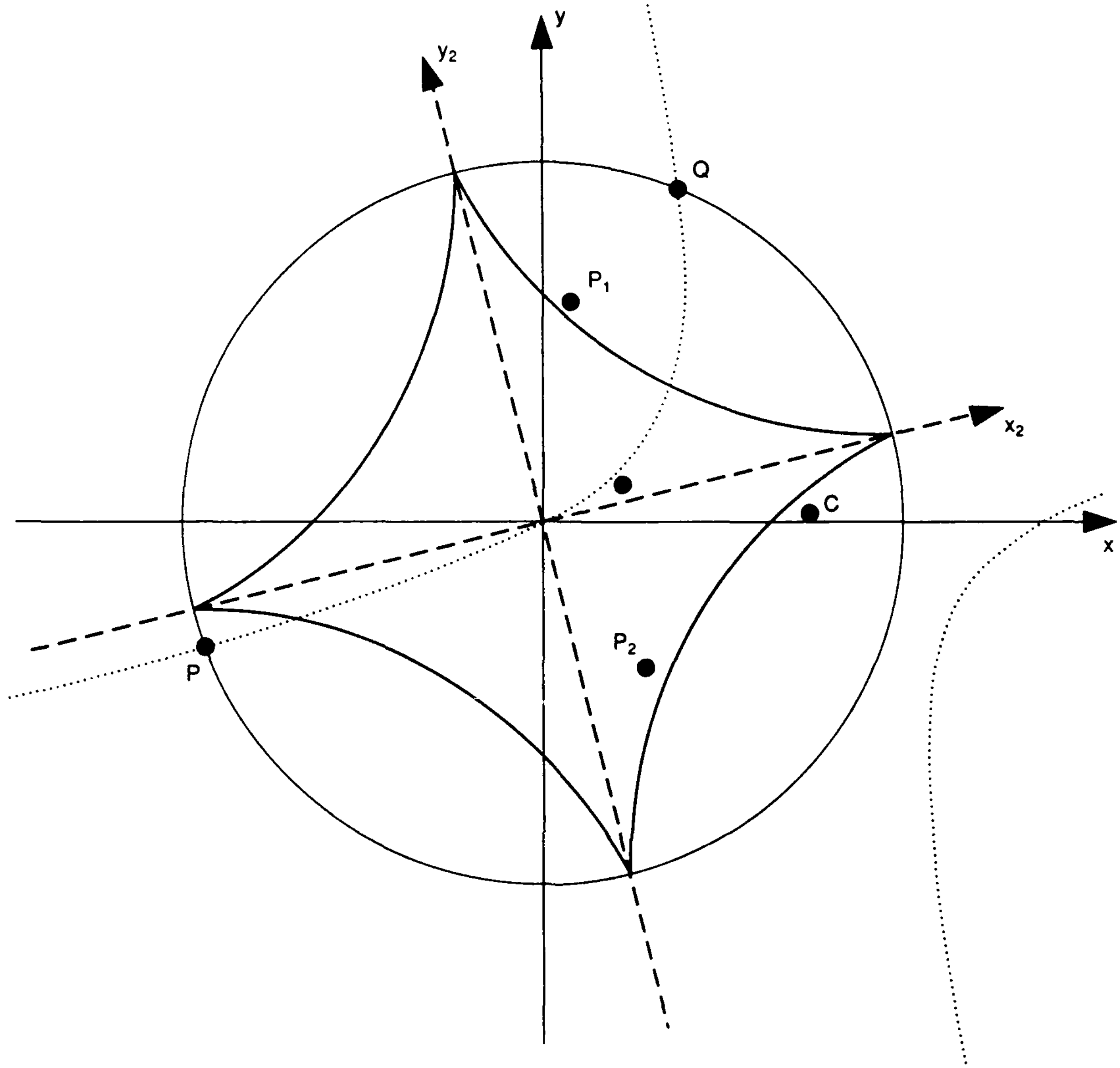
$$x_c^{\frac{2}{3}} + y_c^{\frac{2}{3}} \lesseqgtr r^{\frac{2}{3}} \tag{21}$$

which implies that the problem has four solutions if the center C of the hyperbola is located inside the so-called asteroid, given by

$$x^{\frac{2}{3}} + y^{\frac{2}{3}} = r^{\frac{2}{3}}$$

and two solutions, if C is outside. This is Tychsen's elegant solution to Alhazen's problem. In Figure 3 the asteroid is shown for the same values of the parameters as were used in Figure 2. Note that the above expression for the asteroid refers to the coordinate system C_2.

It is worthwhile to note, that by the above technique the problem has been solved analytically, since the solution of an algebraic quartic equation may be obtained by analytical methods. This is due to the fact that by using Alhazens formulation, no square roots are introduced in Eq. (4), which in turn implies that it is not necessary to square the equation in order to solve it. Squaring immediately raises the degree of the equation above four and thus excludes the possibility of an exact solution. In the next section we shall discuss this problem in further detail.



REFLECTION BY CIRCLE: INPUT DATA

x_1 =	1.000	y_1 =	6.000	x_2 =	3.000	y_2 =	–4.000	R =	10.000
R_1 =	6.083	a_1 =	80.538	R_2 =	5.000	a_2 =	–53.130	v =	13.704

COEFFICIENTS OF POLYNOMIAL

a =	14.000	c =	–27.000	d =	–1.000	e =	2.000

CONSTANTS OF HYPERBOLA

xc =	7.351	yc =	0.108	xm =	2.000	ym =	1.000
xc_1 =	7.168	yc_1 =	–1.637	k =	–11.730		

ANALYTICAL REFLECTION POINTS AT FI = 67.16 199.16
NUMERICAL REFLECTION POINTS AT FI = 67.16 199.16

Figure 3. Tychsen's Asteroid.

2.4 An Alternative Formulation of the Law of Reflection

In most textbooks the law of reflection is usually expressed through Snells laws, namely that the angle of incidence and the angle of reflection are equal and that the planes of incidence and reflection coincide. In Figure 4, S is a source point and F a field point. Using the reflection point R as a reference the position vectors for S and F may be written

$$\overline{RS} = \overline{S_1} = S_1\hat{s}_1$$

$$\overline{RF} = \overline{S_2} = S_2\hat{s}_2$$

Furthermore let

$$\overline{N} = N\hat{n}$$

be a normal vector to the reflecting surface at R. The law of reflection then is usually expressed by the formula

$$\hat{n} \times \hat{s}_1 = \hat{s}_2 \times \hat{n} \tag{22}$$

It is seen that square roots are implicitly involved in Eq. (22) and in order to solve for the coordinates of R it is in general necessary to square it.

Another formulation of the law of reflection may be based directly on Fermat's principle, which states that the optical path length from S via R to F shall be stationary. Referring again to Figure 4, the optical path length, assuming a refractive index of unity, may be written

$$s = s_1 + s_2$$

where s_1 and s_2 are given by square roots. These are preserved after the differentiation implied by Fermat's principle and squaring again must be performed to solve the equation for the coordinates of R.

It seems to be little known that it is indeed possible to formulate the law of reflection without using square roots. Referring to Figure 4, we rewrite Eq. (22) in the form

$$\frac{\overline{N} \times \overline{S}_1}{NS_1} = \frac{\overline{S}_2 \times \overline{N}}{S_2N}$$

and note that

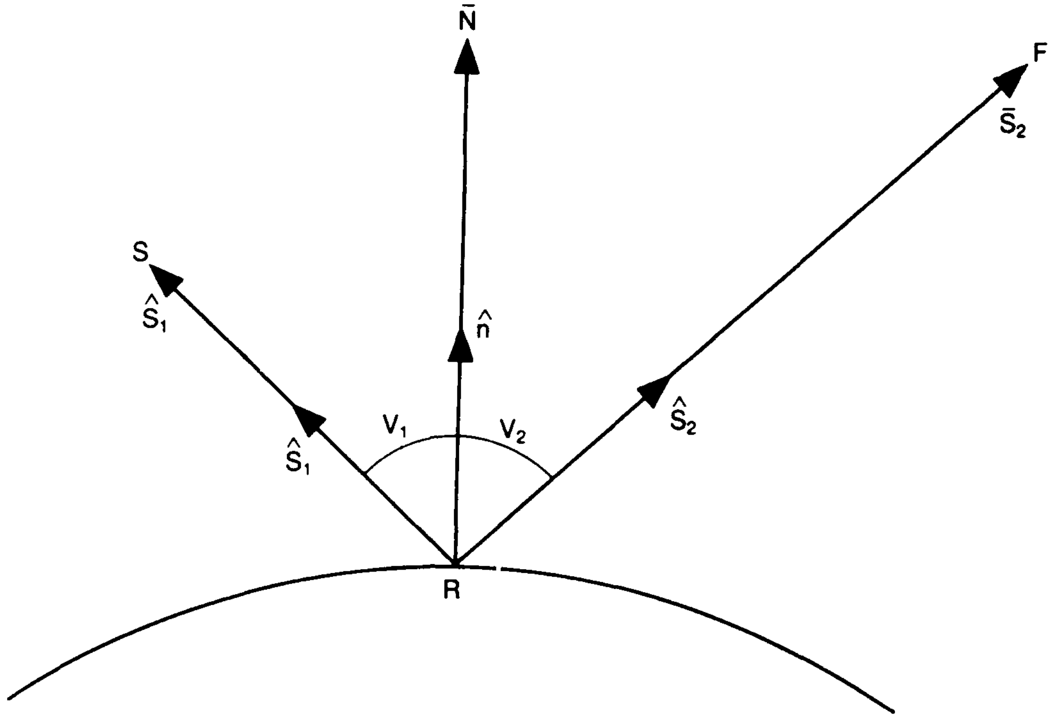


Figure 4. The Law of Reflection.

$$\bar{N}\cdot\bar{S}_1 = NS_1 \cos\, v_1$$
$$\bar{N}\cdot\bar{S}_2 = NS_2 \cos\, v_2$$

where Snell's first law implies that $v_1 = v_2$. Inserting in the above equation then yields

$$\frac{\bar{N}\times\bar{S}_1}{\bar{N}\cdot\bar{S}_1} = \frac{\bar{S}_2\times\bar{N}}{\bar{S}_2\cdot\bar{N}} \tag{23}$$

which is the expression sought. It appears that it contains no square roots and thus is particularly useful for further calculations.

To illustrate the use of the formula, we apply it to Alhazen's problem. We have

$$\bar{S}_1 = (x_1 - x)\hat{x} + (y_1 - y)\hat{y}$$

$$\bar{S}_2 = (x_2 - x)\hat{x} + (y_2 - y)\hat{y}$$

$$\bar{N} = -x\hat{x} - y\hat{y}$$

We find

$$\overline{N}\times\overline{S}_1=(-xy_1+yx_1)\hat{z};\qquad \overline{N}\bullet\overline{S}_1=x^2+y^2-xx_1-yy_1$$

$$\overline{N}\times\overline{S}_2=(-xy_2+yx_2)\hat{z};\qquad \overline{N}\bullet\overline{S}_2=x^2+y^2-xx_2-yy_2$$

Insertion in the law of reflection [Eq. (23)] then immediately yields Alhazen's equation as given by Eq. (4).

If the problem is addressed using the usual law of reflection one obtains

$$\frac{-xy_1+yx_1}{\sqrt{(x_1-x)^2+(y_1-y)^2}}=-\frac{-xy_2+yx_2}{\sqrt{(x_2-x)^2+(y_2-y)^2}} \tag{24}$$

which upon squaring yields

$$\frac{(-xy_1+yx_1)^2}{x^2+y^2+r_1^2-2(xx_1+yy_1)}=\frac{(-xy_2+yx_2)^2}{x^2+y^2+r_2^2-2(xx_2+yy_2)} \tag{25}$$

This is an equation of the fourth degree, that is, one degree higher than Eq. (4).

Very often the solution is based directly on Fermat's principle. Expressing the optical path length as a function of the angle α in Figure 1, we find

$$s=s_1+s_2$$

where

$$s_1=\sqrt{r^2+r_1^2-r(x_1\cos\alpha+y_1\sin\alpha)}$$

$$s_2=\sqrt{r^2+r_2^2-r(x_2\cos\alpha+y_2\sin\alpha)}$$

Differentiating and equating the derivative to zero then yields the equation

$$\frac{-y_1\cos\alpha+x_1\sin\alpha}{s_1}=-\frac{y_2\cos\alpha-x_2\sin\alpha}{s_2} \tag{26}$$

which is identical to Eq. (24). The inexpediency of Eq. (26), in this context, appears when a solution for cos α or sin α is attempted. This leads to an equation of the sixth degree and as it well known such an equation cannot in general be solved analytically.

2.5 The Locus Solution to the Mirror Problem

In this section we shall investigate an alternative solution to Alhazen's problem. Resubstituting $x^2 + y^2$ for r^2 in Eq. (5) it is turned into an equation for the locus of the points for which $v_1 = v_2$, (see Figure 1). The equation becomes

$$2Dx^3 + 2Ex^2y + 2Dxy^2 + 2Ey^3 + Ax^2 + 2Cxy - Ay^2 = 0 \tag{27}$$

that is, an algebraic equation of the third degree. It appears that the origin is contained in the curve and that a maximum of three y-values corresponds to each value of x. The equation may easily be parametrized by introducing

$$y = xt; \qquad t = \tan\alpha \tag{28}$$

which yields

$$x = y_0 \frac{t^2 + 2at - 1}{(t - t_0)(1 + t^2)} \tag{29}$$

where

$$y_0 = -\frac{A}{2E} = \frac{x_1y_2 + x_2y_1}{x_1 + x_2}$$

$$a = -\frac{C}{A} = \frac{x_1x_2 - y_1y_2}{x_1y_2 + x_2y_1}$$

$$t_0 = -\frac{D}{E} = \frac{y_1 + y_2}{x_1 + x_2}.$$

This surprisingly simple equation shows that the curve has a double point at the origin. The corresponding values of the parameter t are found from the equation

$$t^2 + 2at - 1 = 0 \tag{30}$$

with the solutions

$$t_1 = \tan\frac{\alpha_1+\alpha_2}{2}$$

and

$$t_2 = -\cot\frac{\alpha_1+\alpha_2}{2}.$$

Since at the origin t equals the slope of the curve it is seen that the curve intersects itself at right angles. Furthermore it is immediately found that:

(1) The curve intersects the x-axis at

$$x = \frac{y_0}{t_0} = \frac{x_1y_2 + x_2y_1}{y_1+y_2} \qquad \text{for } t = 0$$

(2) The curve intersects the y-axis at

$$y = y_0 = \frac{x_1y_2 + x_2y_1}{x_1+x_2} \qquad \text{for } t = \infty$$

(3) The line

$$y = t_0x = \frac{y_1+y_2}{x_1+x_2}\, x$$

is an asymptote. The asymptote is the line through the origin and the midpoint M between P_1 and P_2.

(4) The points P_1 and P_2 both lie on the curve. This takes some calculations.

(5) Since the curve has only one asymptote and only one double point, it must have a closed loop.

To determine the value of t corresponding to the peak of the loop we note that

$$r = |x|\sqrt{1+t^2}$$

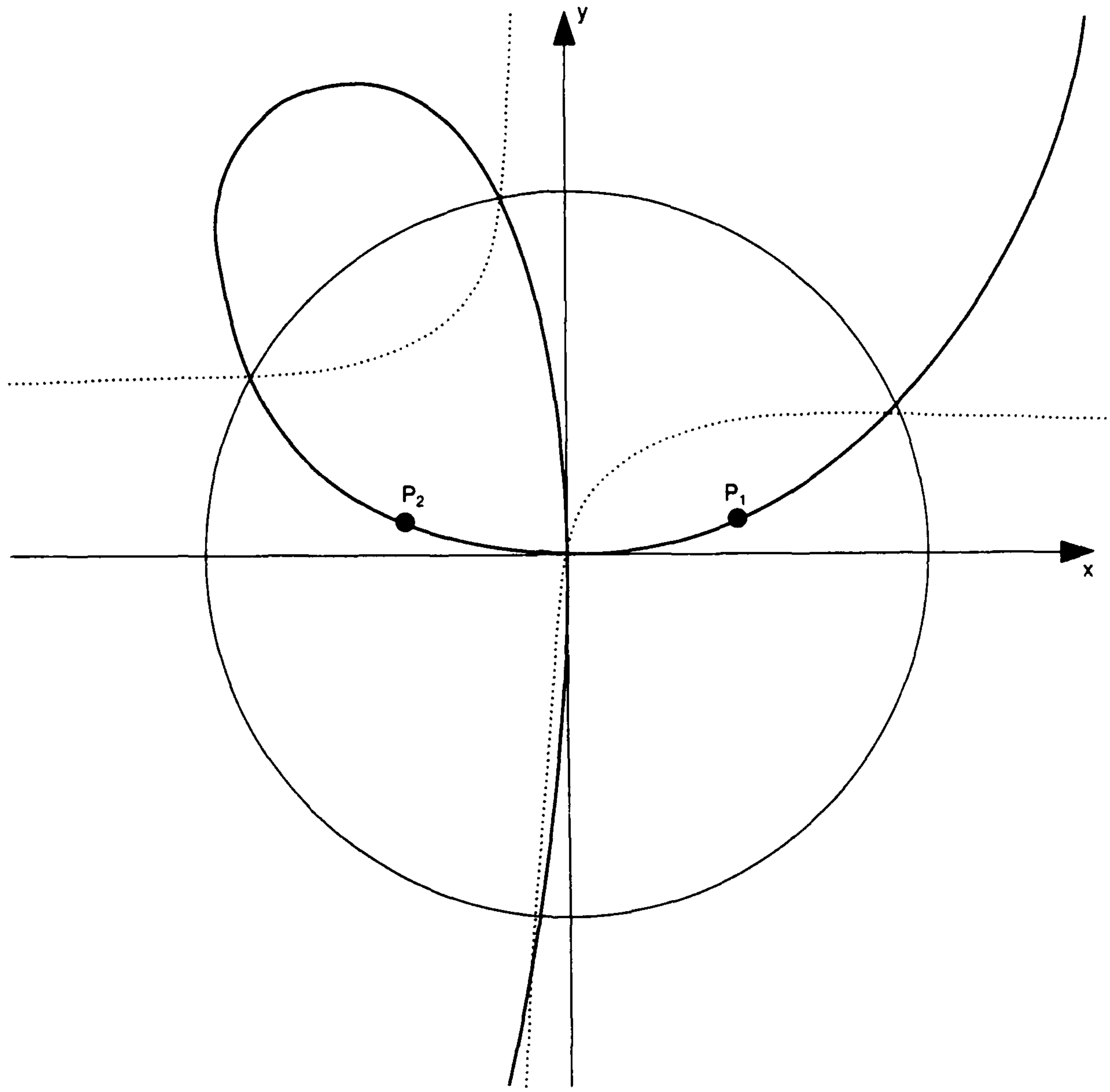
$$= \left| y_0 \frac{t^2 + 2at - 1}{(t - t_0)\sqrt{1+t^2}} \right| \tag{31}$$

Equating the derivative to zero we obtain the third degree equation

$$(2a + t_0)t^3 - 3t^2 + 3t_0 t - 1 = 0 \tag{32}$$

for $t = t_{max}$. This equation can be solved analytically and r_{max} may be determined using Eq. (31). We have then found the following simple alternative rule: The mirror problem has two solutions if $r > r_{max}$ and four solutions if $r < r_{max}$.

In Figure 5 the locus is shown for $(x_1, y_1) = (5, 1)$ and $(x_2, y_2) = (-4.5, 1)$. The asymptote is given by $y = 4x$ and the intersection points with the axes are $(-0.25, 0)$ and $(0, -1)$. It appears that since the radius of the circle is smaller than the height of the loop there are four reflection points.



REFLECTION BY CIRCLE: INPUT DATA

x_1 =	5.000	y_1 =	1.000	x_2 =	–4.500	y_2 =	1.000	R =	10.000
R_1 =	5.099	a_1 =	11.310	R_2 =	4.610	a_2 =	167.471	v =	89.391

COEFFICIENTS OF POLYNOMIAL

a =	0.500	c =	23.500	d =	–1.000	e =	0.250

CONSTANTS OF HYPERBOLA

xc =	–0.973	yc =	4.276	xm =	0.250	ym =	1.000
xc_1 =	4.265	yc_1 =	1.018	k =	4.343		

ANALYTICAL REFLECTION POINTS AT FI =	265.30	150.53	99.75	21.98
NUMERICAL REFLECTION POINTS AT FI =	21.98	99.75	150.53	265.30

Figure 5. The locus (full) and Alhazen's Hyperbola (dotted) for the Mirror Problem.

3. RAY TRACING OF REFLECTED RAYS

3.1 The Sphere Problem

The point of reflection on the convex side of a sphere may be determined analytically by a slight modification of Alhazen's and Tychsen's theory. Since the reflection point must lie in the plane determined by the center of the sphere, the source point, and the field point, the problem is immediately reduced to a plane problem. Thus, as shown in Figure 6, the points P_1 and P_2 are now outside the circle, but by extending the line segments P_1R and P_2R to intersect the x-axis we may again introduce the angles u_1 and u_2 and use Alhazen's technique to determine the hyperbola given by Eq. (6). Thus, formally, the problem again has in general four solutions. As illustrated in Figure 6, one of these corresponds to the actual point of reflection R_1 and another, R_2, to the reflection by the concave side of the circle. The two last solutions, R_3 and R_4, are false and correspond to non-physical solutions where $\tan v_1 = \tan v_2$.

To determine the actual point of reflection, Eq. (20) is solved, again by standard analytical techniques. The angular coordinate α of the reflection point then is given by

$$\cos\alpha = \frac{1-z^2}{1+z^2}; \qquad \sin\alpha = \frac{2z}{1+z^2} \tag{33}$$

The proper root may be determined by the requirement that it should be in the smallest angular region determined by the circle and the two points P_1 and P_2.

3.2 The Cylinder Problem

3.2.1 THE BASIC EQUATION

In this section the theory outlined in Section 2 is applied to the ellipse. The ellipse ray tracing problem arises in connection with the determination of the point of reflection on either side of a circular or an elliptic cylindrical reflector.

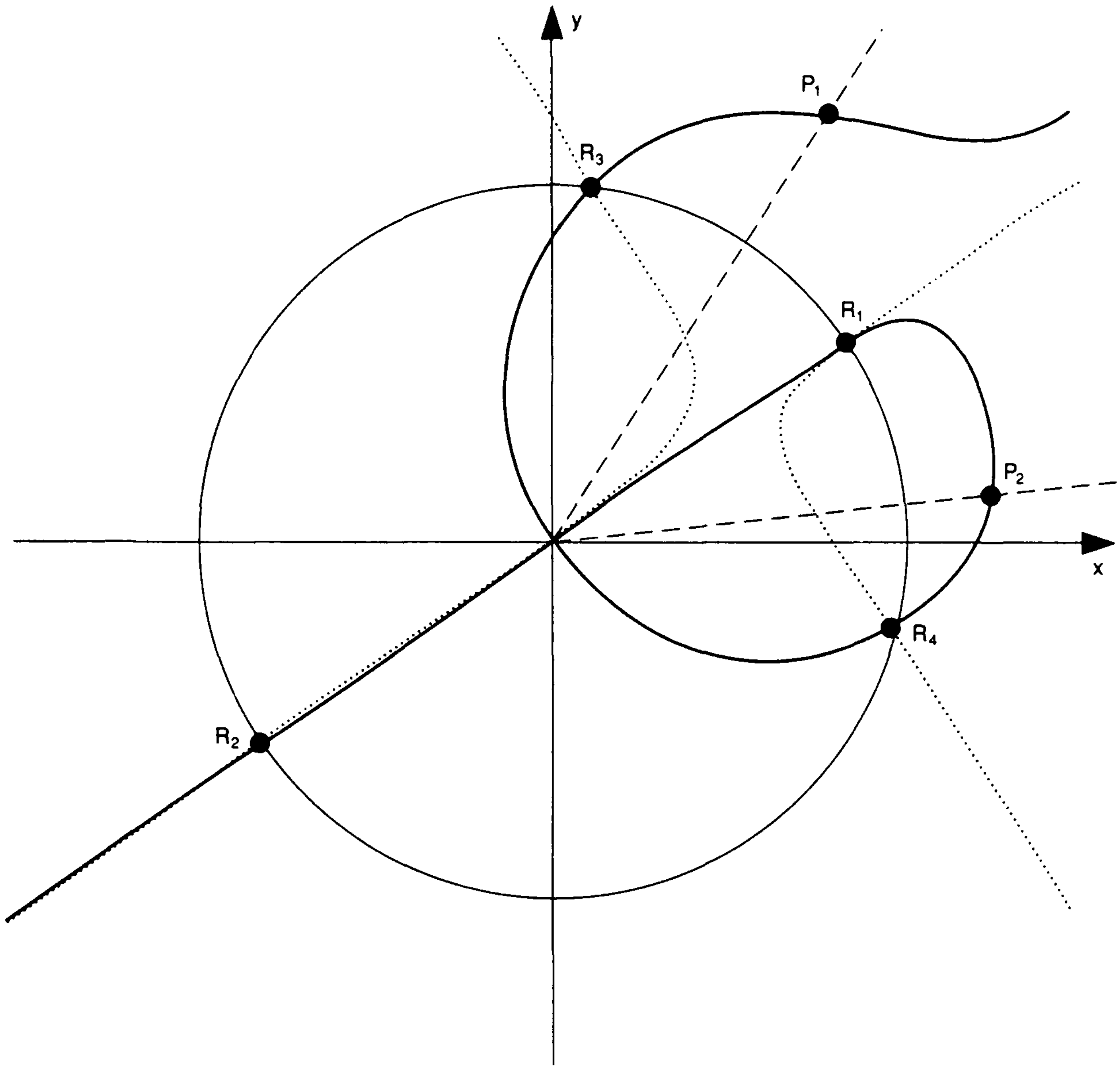
To establish Alhazen's equation for the ellipse we apply the reflection law as given by Eq. (23). Assuming the ellipse is given by

$$b^2x^2 + a^2y^2 = a^2b^2, \tag{34}$$

where a and b are the semi axes, a normal may be written

$$\overline{N} = -\frac{bx}{a}\hat{x} - \frac{ay}{b}\hat{y}.$$

From Eq. (23) we then find the expression



REFLECTION BY CIRCLE: INPUT DATA

x_1 =	8.000	y_1 =	12.000	x_2 =	13.000	y_2 =	3.000	R =	10.000
R_1 =	14.422	a_1 =	56.310	R_2 =	13.342	a_2 =	12.995	v =	34.652

COEFFICIENTS OF POLYNOMIAL

a =	180.000	c =	−68.000	d =	−7.500	e =	10.500

CONSTANTS OF HYPERBOLA

xc =	5.575	yc =	3.727	xm =	10.500	ym =	7.500
xc_1 =	6.705	yc_1 =	−0.104	k =	−0.695		

ANALYTICAL REFLECTION POINTS AT FI = 83.66 32.35 347.51 215.01
NUMERICAL REFLECTION POINTS AT FI = 32.85 215.01

Figure 6. Reflection by sphere. Alhazen's curve is shown dotted, and the Locus by a solid line.

$$\frac{\left(b^2-a^2\right)xy-b^2y_1x+a^2x_1y}{b^2x^2+a^2y^2-b^2x_1x-a^2y_1y}+\frac{\left(b^2-a^2\right)xy-b^2y_2x+a^2x_2y}{b^2x^2+a^2y^2-b^2x_2x-a^2y_2y}=0 \tag{35}$$

To obtain Alhazen's equation corresponding to the ellipse, we introduce the parameter

$$c^4=a^2b^2$$

which we substitute for $b^2x^2 = a^2y^2$ in the above equation. This yields

$$\frac{\left(b^2-a^2\right)xy-b^2y_1x+a^2x_1y}{c^4-b^2x_1x-a^2y_1y}+\frac{\left(b^2-a^2\right)xy-b^2y_2x+a^2x_2y}{c^4-b^2x_2x-a^2y_2y}=0, \tag{36}$$

which may be written

$$a_1x^2y+a_2xy^2+a_3x^2+a_4xy+a_5y^2+c^4\left(a_6xy+a_7x+a_8y\right)=0 \tag{37}$$

where

$$\begin{aligned}
a_1 &= 2b^2\left(a^2-b^2\right)E & a_2 &= 2a^2\left(b^2-a^2\right)D \\
a_3 &= b^4A & a_4 &= 2a^2b^2C \\
a_5 &= -a^4A & a_6 &= 2\left(b^2-a^2\right) \\
a_7 &= 2b^2D & a_8 &= 2a^2E.
\end{aligned} \tag{38}$$

It appears that due to the more complicated geometry of the ellipse the degree of Alhazen's equation has been raised by one, that is, to the third degree, although the terms containing x^3 and y^3 are not present. Furthermore, an additional term a_6xy is introduced, but for $a = b = r$ the equation reduces to Eq. (6) as should be expected.

3.2.2 THE ALHAZEN CURVE

To obtain an expression for Alhazen's curve we rearrange Eq. (37) to yield

$$b_1x^2y+b_2xy^2+b_3x^2+b_4xy+b_5y^2+b_6x+b_7y=0 \tag{39}$$

where

$$
\begin{aligned}
&b_1 = b^2\left(a^2 - b^2\right)\left(x_1 + x_2\right) && b_2 = a^2\left(a^2 - b^2\right)\left(y_1 + y_2\right) \\
&b_3 = b^4\left(x_1 y_2 + x_2 y_1\right) && b_4 = 2a^2 b^2\left(y_1 y_2 - x_1 x_2 + b^2 - a^2\right) \\
&b_5 = -a^4\left(x_1 y^2 + x^2 y_1\right) && b_6 = -a^2 b^4\left(y_1 + y_2\right) \\
&b_7 = a^4 b^2\left(x_1 + x_2\right).
\end{aligned}
\tag{40}
$$

and from which it appears that the curve contains the origin. Since Eq. (39) is a second degree equation in y;

$$
\left(b_2 x + b_5\right) y^2 + \left(b_1 x^2 + b_4 x + b_7\right) y + \left(b_3 x^2 + b_6 x\right) = 0. \tag{41}
$$

in general two values of y correspond to each value of x, and two intervals of x may exist where Alhazen's curve is not defined. Furthermore, the curve intersects the x-axis at $(x_0, 0)$, where

$$
x_0 = -\frac{b_6}{b_3} = a^2 \frac{y_1 + y_2}{x_2 y_1 + x_1 y_2}. \tag{42}
$$

and it appears that in general there is a vertical asymptote at

$$
x = -\frac{b_5}{b_2} = \frac{a^2\left(x_1 y_2 + x_2 y_1\right)}{\left(a^2 - b^2\right)\left(y_1 + y_2\right)}. \tag{43}
$$

Similarly the equation for x

$$
\left(b_1 y + b_3\right) x^2 + \left(b_2 y^2 + b_4 y + b_6\right) x + \left(b_5 y^2 + b_7 y\right) = 0, \tag{44}
$$

shows that the curve intersects the y-axis at $(0, y_0)$, where

$$
y_0 = -\frac{b_7}{b_5} = b^2 \frac{x_1 + x_2}{x_2 y_1 + x_1 y_2}. \tag{45}
$$

and that in general a horizontal asymptote exists for

$$y = -\frac{b_3}{b_1} = \frac{b^2(x_1 y + x_2 y_1)}{(a^2 - b^2)(x_1 + x_2)}. \tag{46}$$

Parametrizing the equation by $y = xt$ we obtain

$$x\left[\left(b_2 t^2 + b_1 t\right)x^2 + \left(b_5 t^2 + b_4 t + b_3\right)x + \left(b_7 t + b_6\right)\right] = 0 \tag{47}$$

from which it appears that in a given direction there are in general two curve points although sectors may exist where Alhazen's curve is not defined. Furthermore, the equation shows that an oblique asymptote appears in the direction given by

$$t = -\frac{b_1}{b_2} = \frac{b^2(x_1 + x_2)}{a^2(y_1 + y_2)}. \tag{48}$$

This asymptote passes through the point (0, q), where

$$q = \frac{b_1 b_5}{b_2^2} - \frac{b_4}{b_2} + \frac{b_3}{b_1}.$$

Thus it is possible to gain a broad idea of the complicated nature of Alhazen's curve, but an analytical solution to the problem seems out of reach in the general case.

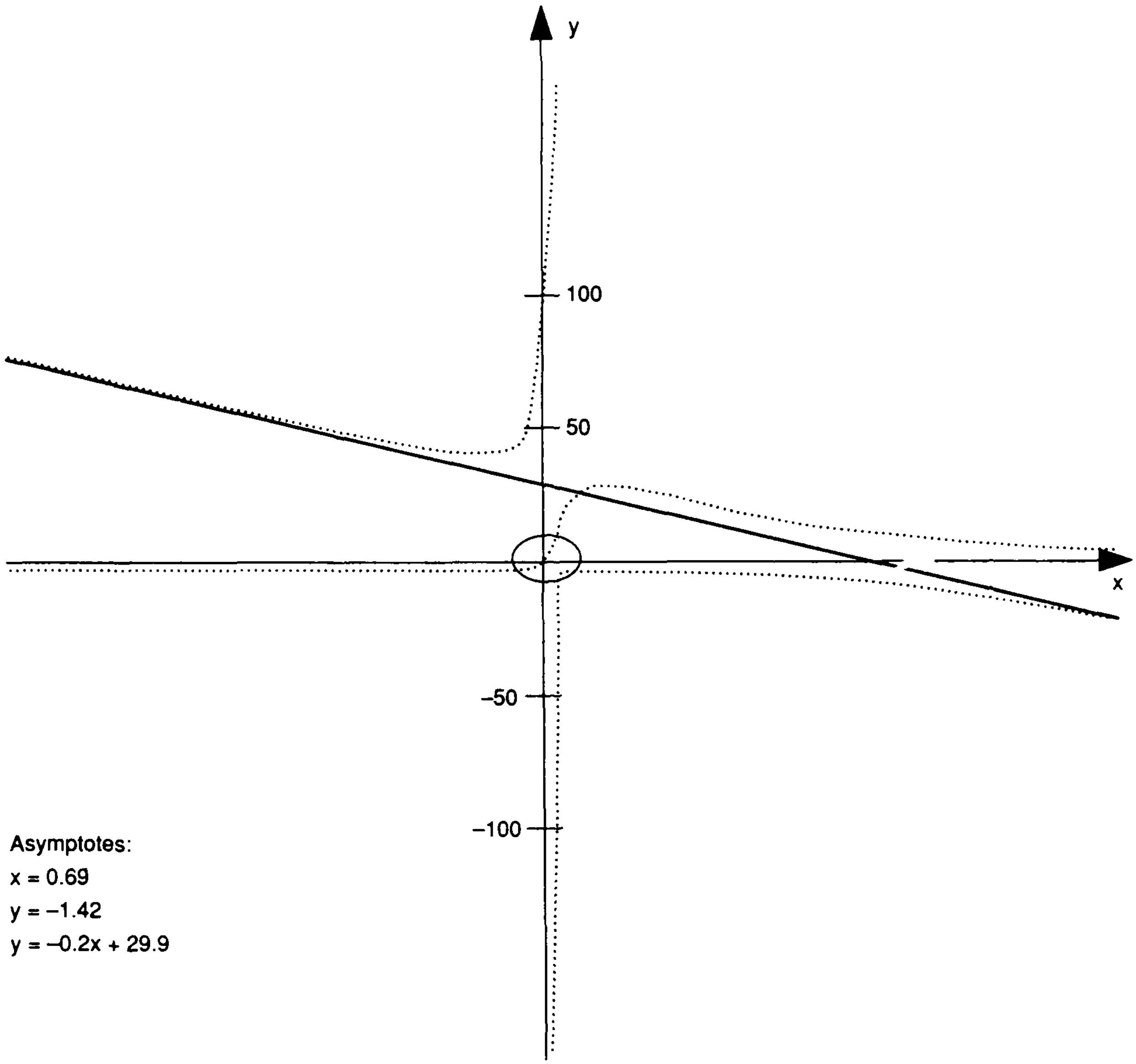
Figure 7 shows an example of Alhazen's curve for the ellipse, namely for $(x_1, y_1) = (5.0, 0.8)$, $(x_2, y_2) = (-4.5, 0.8)$, $a = 10$ and $b = 8$. Although these values are similar to those used for the circle in Figure 5, the general picture is entirely different. It even appears that six reflection points may exist on an ellipse. In the next section we shall look further into this problem.

3.2.3 THE LOCUS

We next consider the locus given by Eq. (37) with

$$c^4 = b^2 x^2 + a^2 y^2.$$

Inserting this and rearranging the equation we obtain



REFLECTION BY ELLIPSE: INPUT DATA

$x_1 =$	5.000	$y_1 =$	0.800	$x_2 =$	-4.500	$y_2 =$	0.800
a =	10.000	b =	8.000				

COEFFICIENTS OF POLYNOMIAL

a =	0.40000E+00	c =	0.23140E+02	d =	-0.80000E+00	e =	0.25000E+00

CONSTANTS OF PSEUDO-LOCUS CURVE

$d_1 =$	-0.65536E+04	$d_2 =$	0.43520E+04	$d_3 =$	-0.44800E+04	$d_4 =$	0.50000E+04
$d_5 =$	0.16384E+04	$d_6 =$	-0.16461E+06	$d_7 =$	-0.40000E+04		
$t_1 =$	-0.41162E+02	$t_2 =$	0.99512E-02	$t_0 =$	0.11405E+01		
$x_0 =$	0.25000E+00	$y_0 =$	0.80000E+00	Q =	-0.14021E+02		

NUMERICAL REFLECTION POINTS AT FI = 84.68 200.16 279.14 331.75

Figure 7. Reflections by an ellipse. Alhazen's curve.

$$c_1x^3y + c_2xy^3 + c_3x^3 + c_4x^2y + c_5xy^2 + c_6y^3 + c_7x^2 + c_8xy + c_9y^2 = 0, \tag{49}$$

where

$$\begin{aligned}
&c_1 = -2b^2(a^2 - b^2) && c_2 = -2a^2(a^2 - b^2) \\
&c_3 = -b^4(y_1 + y_2) && c_4 = b^2(2a^2 - b^2)(x_1 + x_2) \\
&c_5 = a^2(a^2 - 2b^2)(y_1 + y_2) && c_6 = a^4(x_1 + x_2) \\
&c_7 = b^4(x_1y_2 + x_2y_1) && c_8 = 2a^2b^2(y_1y_2 - x_1x_2) \\
&c_9 = -a^4(x_1y_2 + x_2y_1).
\end{aligned} \tag{50}$$

Again due to the ellipse geometry the degree of the locus has been raised by one, that is, to the fourth degree although terms containing x^4 and y^4 are not present. The origin still is on the curve, but in the present case this point is a double point. Since Eq. (49) is a third order equation in y:

$$(c_2x + c_6)y^3 + (c_5x + c_9)y^2 + (c_1x^3 + c_4x^2 + c_8x)y + (c_3x^3 + c_7x^2) = 0, \tag{51}$$

a minimum of one and a maximum of three values of y correspond to each value of x. Apart from the origin, the curve intersects the x-axis at $(x_0, 0)$, where

$$x_o = -\frac{c_7}{c_3} = \frac{x_1y_2 + x_2y_1}{y_1 + y_2} \tag{52}$$

and it appears that in general there is a vertical asymptote at

$$x = -\frac{c_6}{c_2} = \frac{a_2(x_1 + x_2)}{2(a^2 - b^2)} .$$

Similarly the equation for x,

$$(c_1y + c_3)x^3 + (c_4y + c_7)x^2 + (c_2y^3 + c_5y^2 + c_8y)x + (c_6y^3 + c_9y^2) = 0, \tag{53}$$

indicates an intersection of the y-axis at $(0, y_0)$, where

$$y_o = -\frac{c_9}{c_6} = \frac{x_1y_2 + x_2y_1}{x_1 + x_2} \tag{54}$$

and a horizontal asymptote at

$$y = -\frac{c_3}{c_1} = -\frac{b_2(y_1 + y_2)}{2(a^2 - b^2)}. \tag{55}$$

Parametrizing the equation by $y = xt$ we obtain

$$x^2\left[\left(c_2t^3 + c_1t\right)x^2 + \left(c_6t^3 + c_5t^2 + c_4t + c_3\right) + \left(c_9t^2 + c_8t + c_7\right)\right] = 0. \tag{56}$$

Thus in general there are two curve points in each direction t. Since the equation

$$t^2 = -\frac{c_1}{c_2} = -\frac{b^2}{a^2}$$

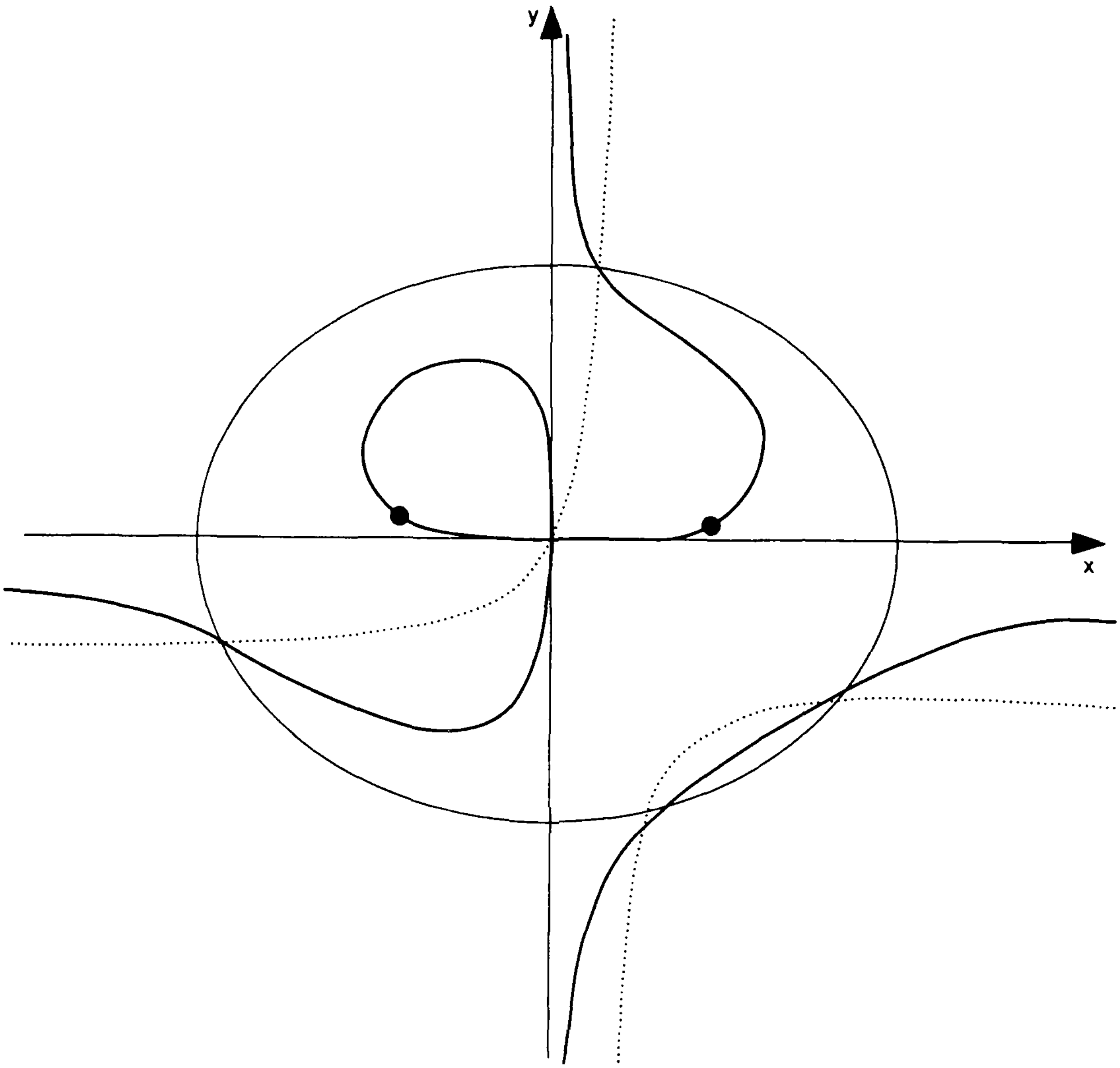
has no real solution, no oblique asymptote is present. Thus in the ellipse case the locus is somewhat simpler than Alhazen's curve. In Figure 8 the locus is shown for the same points P_1 and P_2 as in Figure 6. It appears that the locus has a loop as in the circle case, but also that four reflection points exist although the ellipse does not intersect the loop but rather the additional branch of the curve now present.

We next return to Eq. (49) and note that the coefficients c_1 and c_2 do not depend on the coordinates of P_1 and P_2. The two first terms of Eq. (49) thus may be written

$$c_1x^3y + c_2xy^3 = -2\left(a^2 - b^2\right)\left(b^2x^2 + a^2y^2\right)xy \tag{57}$$

We may now use Alhazen's argument that we are interested only in the points of intersection between the locus and the ellipse and introduce Eq. (34) for the ellipse

$$b^2x^2 + a^2y^2 = a^2b^2 = c^4$$



REFLECTION BY ELLIPSE: INPUT DATA

x_1 =	5.000	y_1 =	0.800	x_2 =	−4.500	y_2 =	0.800
a =	10.000	b =	8.000				

COEFFICIENTS OF POLYNOMIAL

a =	0.40000E+00	c =	0.23140E+02	d =	−0.80000E+00	e =	0.25000E+00

CONSTANTS OF PSEUDO-LOCUS CURVE

d_1 =	−0.65536E+04	d_2 =	0.43520E+04	d_3 =	−0.44800E+04	d_4 =	0.50000E+04
d_5 =	0.16384E+04	d_6 =	−0.16461E+06	d_7 =	−0.40000E+04		
t_1 =	−0.41162E+02	t_2 =	0.99512E−02	t_0 =	0.11405E+01		
x_0 =	0.25000E+00	y_0 =	0.80000E+00	Q =	−0.14021E+02		

NUMERICAL REFLECTION POINTS AT FI = 84.68 200.16 279.14 331.75

Figure 8. The locus in the ellipse case.

and obtain

$$c_1x^3y + c_2xy^3 = -2(a^2 - b^2)c^4xy. \tag{58}$$

Introducing this in Eq. (49) yields what we shall call the pseudo-locus. Rearranging the terms we find

$$d_1x^3 + d_2x^2y + d_3xy^2 + d_4y^3 + d_5x^2 + d_6xy + d_7y^2 = 0 \tag{59}$$

where

$$\begin{aligned} &d_1 = -b^4(y_1 + y_2) && d_2 = b^2(2a^2 - b^2)(x_1 + x_2) \\ &d_3 = a^2(a^2 - 2b^2)(y_1 + y_2) && d_4 = a^4(x_1 + x_2) \\ &d_5 = b^4(x_1y_2 + x_2y_1); && d_6 = 2a^2b^2(y_1y_2 - x_1x_2 - a^2 + b^2) \\ &d_7 = -a^4(x_1y_2 + x_2y_1). && \end{aligned} \tag{60}$$

A comparison with Eq. (27) for the locus in the circle case shows that the two equations are formally identical. In the circle case, however, $d_1 = d_3$, $d_2 = d_4$ and $d_5 = -d_7$, so the geometry of the ellipse disturbs the symmetry of the locus equation and prevents a thorough analysis of the curve as in the circle case.

Rearranging the terms in descending powers of y yields

$$d_4y^3 + (d_3x + d_7)y^2 + (d_2x^2 + d_6x)y + d_1x^3 + d_5x^2 = 0, \tag{61}$$

which shows that in general there are three values of y for each value of x. The curve intersects the x-axis at the origin and at the point $(x_0, 0)$, where

$$x_0 = -\frac{d_5}{d_1} = \frac{x_1y_2 + x_2y_1}{y_1 + y_2} \tag{62}$$

and it has no vertical asymptotes. The analogous equation in x,

$$d_1x^3 + (d_2y + d_5)x^2 + (d_3y^2 + d_6y)x + d_4y^3 + d_7y^2 = 0, \tag{63}$$

shows that in general three values of x correspond to each value of y. Furthermore the curve intersects the y-axis at $(0, y_0)$, where

$$y_0 = -\frac{d_7}{d_4} = \frac{x_1y_2 + x_2y_1}{x_1 + x_2} \tag{64}$$

and it has no horizontal asymptotes. Parametrizing the equation by $y = xt$ and solving for x, we get

$$x = -\frac{d_7t^2 + d_6t + d_5}{d_4t^3 + d_3t^2 + d_2t + d_1}. \tag{65}$$

This equation reveals several interesting facts. First, it shows that there is only one curvepoint in any direction. Next, since the discriminator of the numerator is always positive, the curve always passes through the origin and this point is a double point. Furthermore, at the origin the tangents are no longer at right angles to each other. Finally, it appears that since the equation

$$d_4t^3 + d_3t^2 + d_2t + d_1 = 0$$

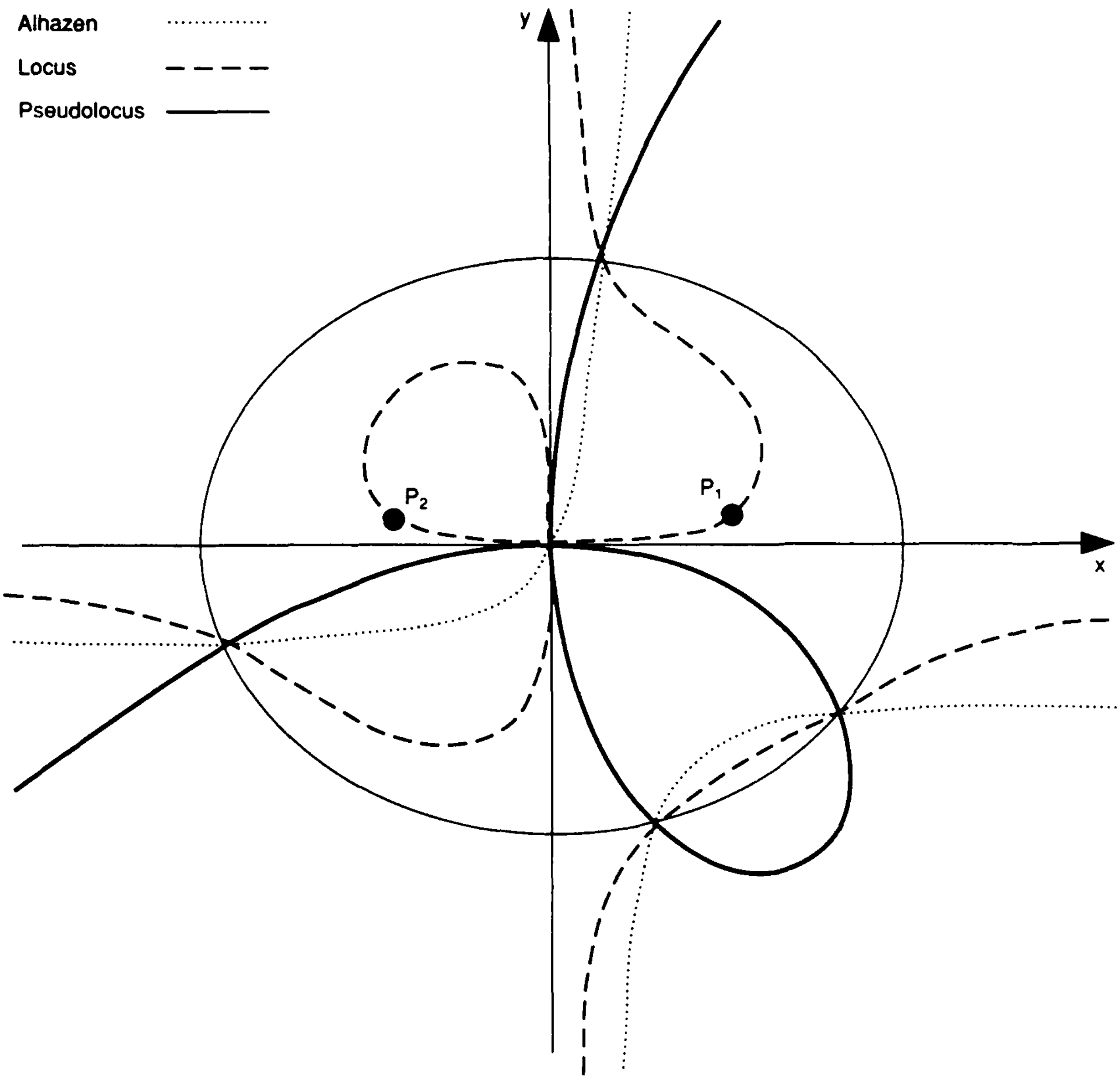
always has one real solution $t = t_0$ there will always be at least one oblique asymptote with a slope determined by t_0. It is easily shown that this asymptote passes through the point $(0, q)$, where

$$q = \frac{d_7t_0^2 + d_6t_0 + d_5}{3d_4t_0^2 + 2d_3t_0 + d_2}.$$

Since the presence of more than one oblique asymptote is in general incompatible with the fact that the curve has only one point in each direction we conclude that the pseudo-locus has a closed loop. This implies that there is a maximum of four intersection points with the ellipse. Furthermore we may conclude that the two branches of Alhazens curve not passing through the origin cannot both intersect the ellipse (see Figure 7). Figure 9 shows the pseudo-locus for the same points P_1 and P_2 as in Figures 7 and 8.

The above results suggest an attempt to solve the problem by using the multiplicative mapping that maps the ellipse onto the circle $x^2 + y^2 = a^2$.

Application of this mapping to the pseudo-locus results in a geometrical configuration very similar to that of Figure 5 for the circle. It turns out, however, that the equation analogous to Eq. (27) is an algebraic equation of a degree higher than four. It is also worthwhile to note that although the points of intersection between the ellipse and the pseudo-locus represent reflection points on the ellipse, the points of intersection between the mapped pseudo-locus and the circle $x^2 + y^2 = a^2$ do not in general correspond to reflection points on the circle. Neither the original



REFLECTION BY ELLIPSE: INPUT DATA

$x_1 =$ 5.000 $y_1 =$ 0.800 $x_2 =$ −4.500 $y_2 =$ 0.800

a = 10.000 b = 8.000

COEFFICIENTS OF POLYNOMIAL

a =0.40000E+00 c = 0.23140E+02 d = −0.80000E+00 e = 0.25000E+00

CONSTANTS OF PSEUDO-LOCUS CURVE

$d_1 =$ −0.65536E+04 $d_2 =$ 0.43520E+04 $d_3 =$ −0.44800E+04 $d_4 =$ 0.50000E+04

$d_5 =$ 0.16384E+04 $d_6 =$ −0.16461E+06 $d_7 =$ −0.40000E+04

$t_1 =$ −0.41162E+02 $t_2 =$ 0.99512E−02 $t_0 =$ 0.11405E+01

$x_0 =$ 0.25000E+00 $y_0 =$ 0.80000E+00 Q = −0.14021E+02

NUMERICAL REFLECTION POINTS AT FI = 84.68 200.16 279.14 331.75

Figure 9. Reflection by an Ellipse. Alhazen's curve (dotted), and the Pseudo-locus (solid).

points P_1 and P_2 nor the points obtained by applying the mapping to these points are reflection points. On the other hand the mapping may be used for an a priori determination of the number of reflection points.

4. TRACING OF DIFFRACTED RAYS

4.1 The Basic Equation

The generalization of Alhazen's problem to the ray tracing problem associated with diffraction by a circular edge is based on the similarities between the laws of reflection and diffraction. These laws may be written

$$\hat{s}_1 \times \hat{n} + \hat{s}_2 \times \hat{n} = 0 \qquad \text{and} \qquad \hat{s}_1 \bullet \hat{t} + \hat{s}_2 \bullet \hat{t} = 0$$

where $\hat{s}_1$ is a unit vector pointing from the reflection– or the diffraction point– to P_1, $\hat{s}_2$ points from the same point to P_2, $\hat{n}$ is a unit vector normal to the circle (in the plane of the circle), while $\hat{t}$ is a unit vector tangent to the circle. The reflection law states that the angles of incidence and reflection, v_1 and v_2, are equal and that the planes of incidence and reflection coincide. The diffraction law states that the angles of incidence and diffraction, v_1 and v_2 are equal and that the planes of incidence and diffraction do not in general coincide. Since Alhazen's theory relies on the angular relation $v_1 = v_2$ and not explicity on the vectorial nature of the reflection law, it may be generalized to include the diffraction problem also.

In Figure 10, the circular edge is given by $x^2 + y^2 = r^2$ and the point D, with coordinates $(x_D, y_D, z_D) = (r \cos \alpha, r \sin \alpha, 0)$, is assumed to be a point of diffraction. At this point we introduce a rectangular coordinate system (x_1, y_1, z_1) with the z_1-axis tangential to the circle and the y_1-axis perpendicular to the xy-plane. The z_1-axis thus is the axis of the Keller cone with its apex at D. The cone intersects the xy-plane in the line l (and l_1) and we denote the angle between this line and the line OD by v. The apex angle of the cone then is $2(90 - v)$ and the equation of the cone becomes

$$x_1^2 + y_1^2 - \cot^2 v z_1 = 0 \tag{66}$$

To find the equation of the Keller cone in the xyz-system we apply the transformation

$$\begin{Bmatrix} x_1 \\ y_1 \\ z_1 \end{Bmatrix} = \begin{Bmatrix} \cos\alpha & \sin\alpha & 0 \\ 0 & 0 & 1 \\ -\sin\alpha & \cos\alpha & 0 \end{Bmatrix} + \begin{Bmatrix} x - x_D \\ y - y_D \\ z \end{Bmatrix} \tag{67}$$

and find after some reductions

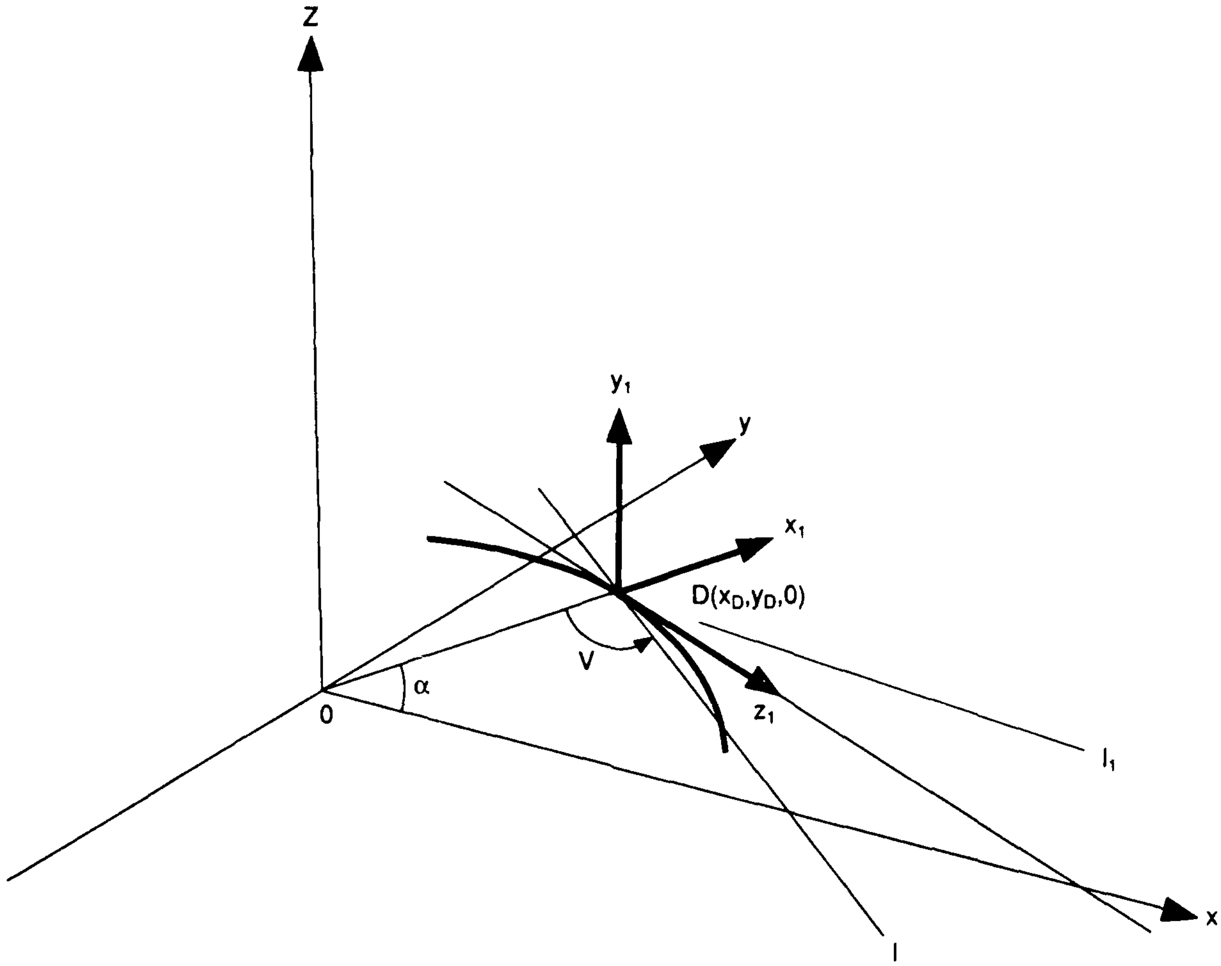


Figure 10. The Keller Cone.

$$\tan^2 v = \frac{(yx_D - xy_D)^2}{(r^2 - xx_D - yy_D)^2 + r^2 z^2} \tag{68}$$

We next introduce the two points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ and the corresponding angles v_1 and v_2. Since these points must lie on the cone [Eq. (66)] we find from the law of diffraction that $v_1 = v_2$, or

$$\tan^2 v_1 = \tan^2 v_2 \tag{69}$$

and that

$$\frac{(y_1x_D - x_1y_D)^2}{\left(r^2 - x_1x_D - y_1y_D\right)^2 + r^2z_1^2} = \frac{(y_2x_D - x_2y_D)^2}{\left(r^2 - x_2x_D - y_2y_D\right)^2 + r^2z_1^2} \tag{70}$$

It is noted, that for $z_1 = z_2 = 0$ the formula is reduced to Eq. (4), however squared. To investigate the formula derived in more detail, we introduce the following functions

$$\begin{aligned} f_1(x,y) &= y_1x - x_1y & f_2(x,y) &= y_2x - x_2y \\ g_1(x,y) &= r^2 - x_1x - y_1y & g_2(x,y) &= r^2 - x_2x - y_2y. \end{aligned} \tag{71}$$

Neglecting index D we may then write Eq. (69) in the form

$$\frac{f_1^2(x,y)}{g_1^2(x,y) + r^2z_1^2} = \frac{f_2^2(x,y)}{g_2^2(x,y) + r^2z_2^2} \tag{72}$$

or, after some reductions

$$(f_1g_2 + f_2g_1)(f_1g_2 - f_2g_1) + r^2\left(f_1^2z_2^2 - f_2^2z_1^2\right) = 0. \tag{73}$$

Next it appears from Eqs. (5) and (6) that

$$f_1g_2 + f_2g_1 = -\left[Ax^2 - Ay^2 + 2Cxy + r^2(2Dx + 2Ey)\right] \tag{74}$$

while

$$f_1g_2 - f_2g_1 = r^2(2Fx + 2Gy + K) \tag{75}$$

where

$$F = 0.5(y_1 - y_2); \quad G = -0.5(x_1 - x_2); \quad K = x_1y_2 - x_2y_1 \tag{76}$$

Furthermore the last term of the equation may be written

$$r^2\left(f_1^2 z_2^2 - f_2^2 z_1^2\right) = -r^2\left(Lx^2 + 2Mxy + Ny^2\right) \tag{77}$$

where

$$L = y_2^2 z_1^2 - y_1^2 z_2^2$$

$$M = x_1 y_1 z_2^2 - x_2 y_2 z_1^2$$

$$N = x_2^2 z_1^2 - x_2^2 z_2^2$$

such that Eq. (72) may be written

$$\left(Ax^2 - Ay^2 + 2Cxy + r^2(2Dx + 2Ey)\right)(2Fx + 2Gy + K) + Lx^2 + Mxy + Ny^2 = 0. \tag{78}$$

It thus appears that the curves, which in Alhazens problem were conic sections, in the case of diffraction are algebraic curves of the third degree. Performing the multiplications and rearranging the terms we obtain

$$\begin{aligned} &a_1x^3 + a_2x^2y + a_3xy^2 + a_4y^3 + a_5x^2 + a_6xy + a_7y^2 \\ &+ r^2\left(a_8x^2 + a_9xy + a_{10}y^2 + a_{11}x + a_{12}y\right) = 0, \end{aligned} \tag{79}$$

where the coefficients are given by

$$\begin{array}{lll} a_1 = 2AF & a_2 = 4CF + 2AG & a_3 = -2AF + 4GC \\ a_4 = -2AG & a_5 = AK + L & a_6 = 2CK + 2M \\ a_7 = -AK + N & a_8 = 4DF & a_9 = 4EF + 4DG \\ a_{10} = 4GE & a_{11} = 2DK & a_{12} = 2EK. \end{array} \tag{80}$$

where the various constants are given by Eqs. (7), (74) and (77).

4.2 The Alhazen Curve

Rearranging the terms of Eq. (79) we get

$$b_1x^3 + b_2x^2y + b_3xy^2 + b_4y^3 + b_5x^2 + b_6xy + b_7y^2 + b_8x + b_9y = 0 \tag{81}$$

where

$$b_1 = a_1 \qquad b_2 = a_2 \qquad b_3 = a_3$$

$$b_4 = a_4 \qquad b_5 = a_5 + r^2a_9 \qquad b_6 = a_6 + r^2a_9$$

$$b_7 = a_7 + r^2a_{10} \qquad b_8 = r^2a_{11} \qquad b_9 = r^2a_{12}.$$

From the y-equation

$$b_4y^3 + (b_3x + b_7)y^2 + \left(b_2x^2 + b_6x + b_9\right)y + b_1x^3 + b_5x^2 + b_8x = 0, \tag{82}$$

it appears that there is a maximum of three y-values for each value of x. There is no vertical asymptote and the curve intersects the x-axis where

$$x\left(b_1x^2 + b_5x + b_8\right) = 0,$$

that is, at the origin plus possibly at two more points. Similarly the x-equation

$$b_1x^3 + (b_2y + b_5)x^2 + \left(b_3y^2 + b_6y + b_8\right)x + \left(b_4y^3 + b_5y^2 + b_9y\right) = 0 \tag{83}$$

shows that there are in general three x-values for each value of y. There is no horizontal asymptote and the curve intersects the y-axis where

$$b_4y^2 + b_5y + b_9 = 0,$$

that is, apart from the origin at two points at the most. Finally the t-equation

$$x\left[\left(b_4t^3 + b_3t^2 + b_2t + b_1\right)x^2 + \left(b_7t^2 + b_6t + b_5\right)x + b_9t + b_5\right] = 0 \tag{84}$$

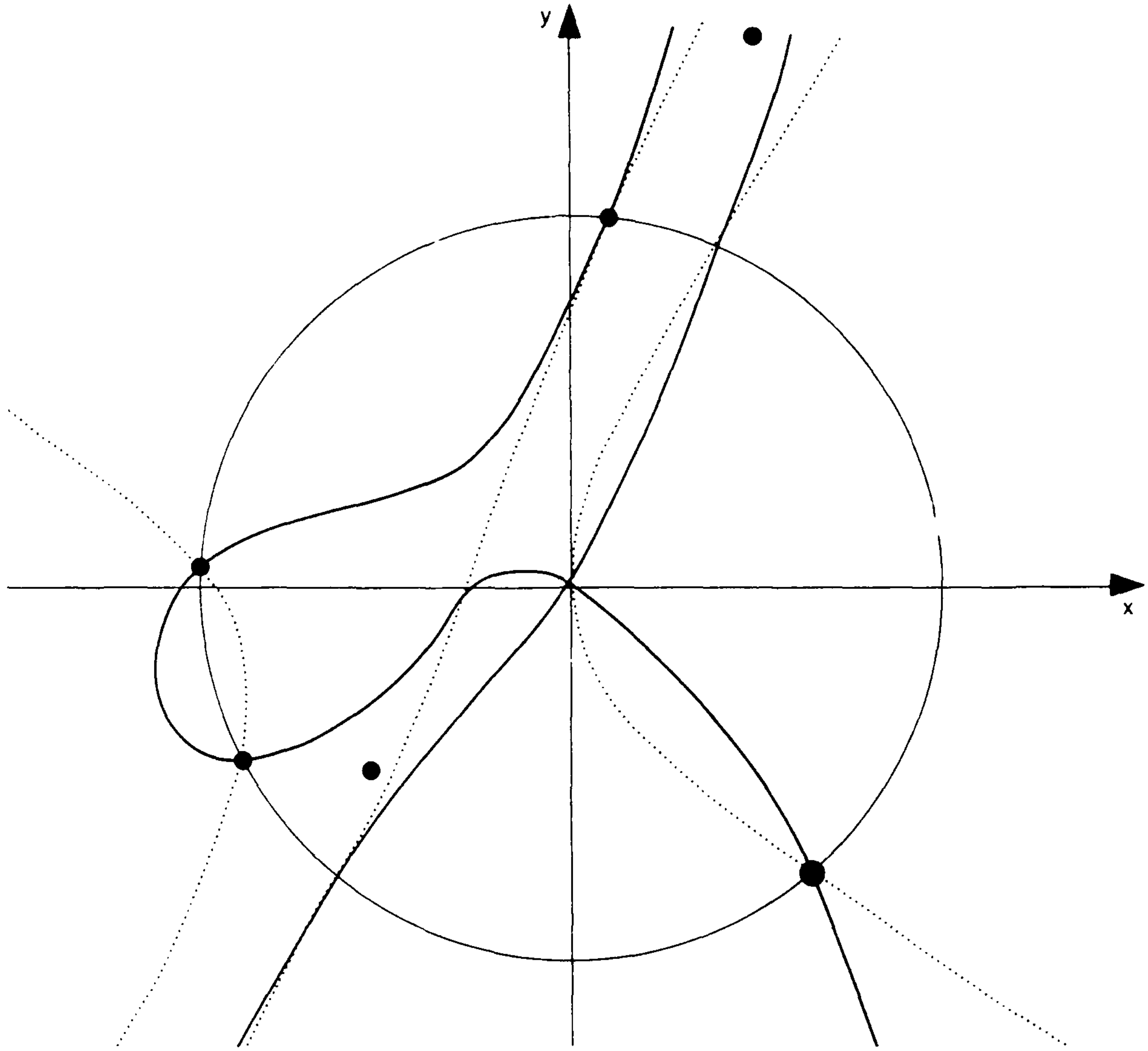
shows that in general there are two curve points in each direction and that the origin is not a double point. Directions may exist where the curve is not defined and finally in general there are three oblique asymptotes.

The fact that there are three oblique asymptotes suggests that six diffraction points may be possible. We can, however, eliminate two of these by the following argument. The two points P_1 and P_2 must not be located on the same Keller cone, but rather, one point must be on each of the two cones given by Eq. (70). This implies that the points must not be on the same side of a plane perpendicular to the axis of the Keller cone at the point of diffraction, which means that their projections (x_1, y_1) and (x_2, y_2) must not be on the same side of the line OD. Assuming coordinates (x_D, y_D) for the diffraction points leads to the condition

$$(y_D x_1 - x_D y_1)(y_D x_2 - x_D y_2) < 0. \tag{85}$$

The condition may also be formulated as follows: A diffraction point can never be on the part of the circular edge that corresponds to the neighboring angles to the angle defined by the origin and the radii to the projections of P_1 and P_2 on the plane of the edge.

Figure 11 shows an example of Alhazen's curve for $(x_1, y_1, z_1) = (5, 15, 10)$, $(x_2, y_2, z_2) = (-5, -5, 3)$ and a circle with $r = 10$. It is noted that there are six points of intersection with the circle. In Figure 12 the condition Eq. (85) has been applied. Two of the intersection points are now excluded and the remaining four points correspond to the diffraction points found by a numerical search routine.



DIFFRACTION BY CIRCLE: INPUT DATA

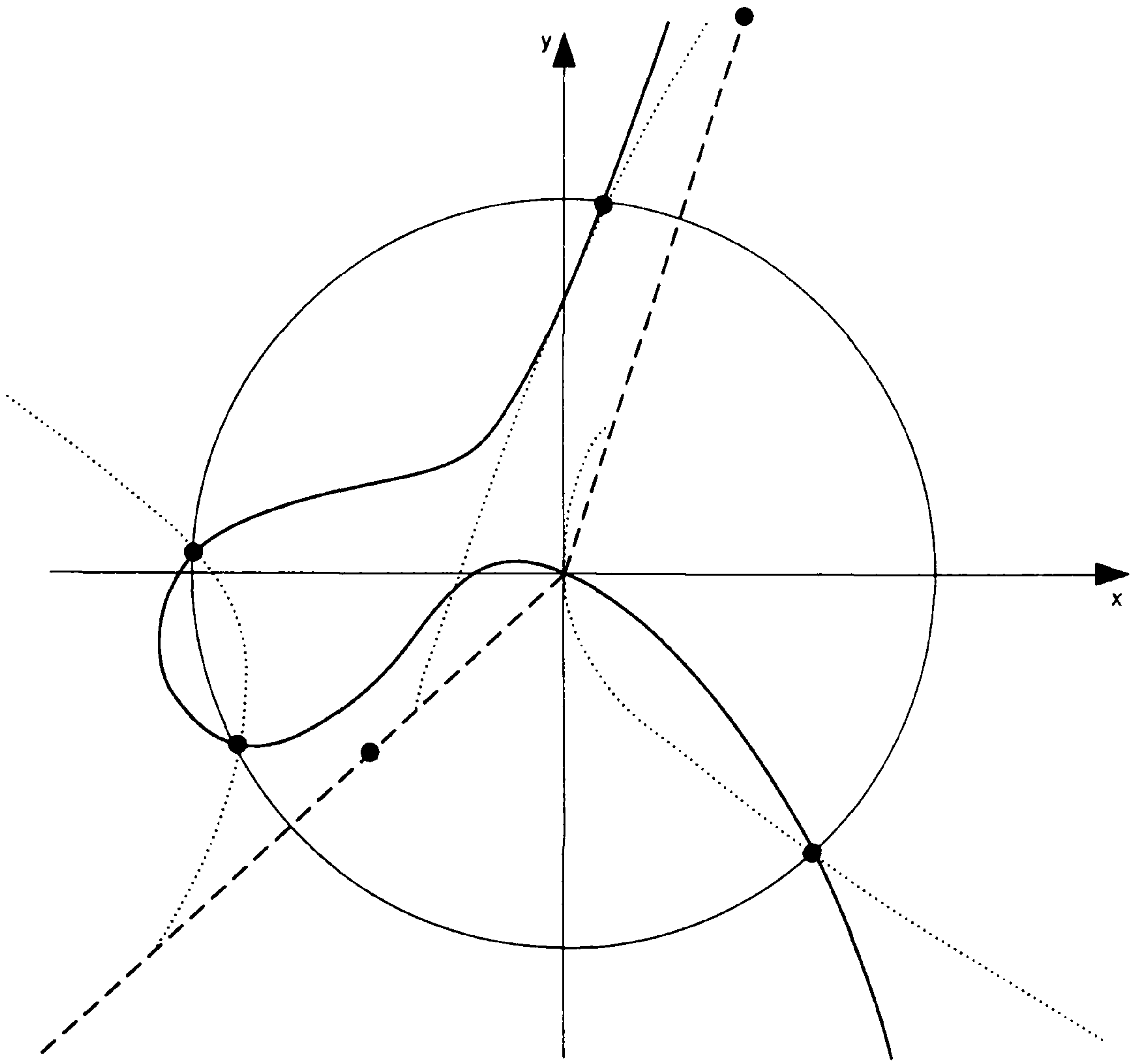
x_1 =	5.000	y_1 =	15.000	z_1 =	10.000
x_2 =	−5.000	y_2 =	−5.000	z_2 =	3.000

DIFFRACTION POINTS AT FI = 82.81 178.06 208.86 313.42

COEFFICIENTS OF POLYNOMIALS

a =	−0.10000E+03	c =	−0.50000E+02	d =	−0.50000E+01	e =	0.00000E+00
f =	0.10000E+02	g =	−0.50000E+01	k =	0.50000E+02		
l =	0.47500E+03	m =	−0.18250E+04	n =	0.22750E+04		
a_1 =	−0.20000E+04	a_2 =	−0.10000E+04	a_3 =	0.30000E+04	a_4 =	−0.10000E+04
a_5 =	−0.45250E+04	a_6 =	−0.86500E+04	a_7 =	0.72750E+04	a_8 =	−0.20000E+03
a_9 =	0.10000E+03	a_{10} =	0.00000E+00	a_{11} =	−0.50000E+03	a_{12} =	0.00000E+00
b_1 =	−0.20000E+04	b_2 =	−0.10000E+04	b_3 =	0.30000E+04		
b_4 =	−0.10000E+04	b_5 =	−0.24525E+05	b_6 =	0.13500E+04		
b_7 =	0.72750E+04	b_8 =	−0.50000E+05	b_9 =	0.00000E+00		

Figure 11. Diffraction by a circular edge. Alhazen's curve and the locus



DIFFRACTION BY CIRCLE: INPUT DATA

x_1 = 5.000	y_1 = 15.000	z_1 = 10.000
x_2 = –5.000	y_2 = –5.000	z_2 = 3.000

DIFFRACTION POINTS AT FI = 82.81 178.06 208.86 313.42

COEFFICIENTS OF POLYNOMIALS

a = –0.10000E+03	c = –0.50000E+02	d = –0.50000E+01	e = 0.00000E+00
f = 0.10000E+02	g = –0.50000E+01	k = 0.50000E+02	
l = 0.47500E+03	m = –0.18250E+04	n = 0.22750E+04	
a_1 = –0.20000E+04	a_2 = –0.10000E+04	a_3 = 0.30000E+04	a_4 = –0.10000E+04
a_5 = –0.45250E+04	a_6 = –0.86500E+04	a_7 = 0.72750E+04	a_8 = –0.20000E+03
a_9 = 0.10000E+03	a_{10} = 0.00000E+00	a_{11} = –0.50000E+03	a_{12} - 0.00000E+00
b_1 = –0.20000E+04	b_2 = –0.10000E+04	b_3 = 0.30000E+04	
b_4 = –0.10000E+04	b_5 = –0.24525E+05	b_6 = 0.13500E+04	
b_7 = 0.72750E+04	b_8 =—–0.50000E+05	b_9 = 0.00000E+00	

Figure 12. Alhazen's curve and locus with separation condition.

4.3 The Locus

Finally we consider the locus for diffraction by a circular edge. Substituting $x^2 + y^2$ for r^2 in Eq. (79) and rearranging the terms we find

$$\begin{aligned} & c_5 y^4 + (c_4 x + c_9) y^3 + \left(c_3 x^2 + c_8 x + c_{12}\right) y^2 \\ & + \left(c_2 x^3 + c_7 x^2 + c_{11} x\right) y + c_1 x^4 + c_6 x^3 + c_{10} x^2 = 0 \end{aligned} \tag{86}$$

where

$$\begin{array}{lll} c_1 = a_8 & c_2 = a_9 & c_3 = a_8 + a_{10} \\ c_4 = a_9 & c_5 = a_{10} & c_6 = a_1 + a_{11} \\ c_7 = a_2 + a_{12} & c_8 = a_3 + a_{11} & c_9 = a_4 + a_{12} \\ c_{10} = a_5 & c_{11} = a_6 & c_{12} = a_7 . \end{array} \tag{87}$$

Eq. (86) contains all terms of a general fourth degree polynomial in two variables except the first degree terms and the constant term, which implies that the curve contains the origin. From the y-equation

$$\begin{aligned} & c_5 y^4 + (c_4 x + c_9) y^3 + \left(c_3 x^2 + c_8 x + c_{12}\right) y^2 \\ & + \left(c_2 x^3 + c_7 x^2 + c_{11} x\right) y + c_1 x^4 + c_6 x^3 + c_{10} x^2 = 0 \end{aligned} \tag{88}$$

it appears that there is a maximum of four y-values for each value of x, that no vertical asymptote is present and that the curve intersects the x-axis where

$$x^2\left(c_1 x^2 + c_6 x + c_{10}\right) = 0,$$

that is, at the origin plus possibly two more points. Similarly the x-equation

$$\begin{aligned} & c_1 x^4 + (c_2 y + c_6) x^3 + \left(c_3 y^2 + c_7 y + c_{10}\right) x^2 \\ & + \left(c_4 y^3 + c_8 y^2 + c_{11} y\right) x + c_5 y^4 + c_9 y^3 + c_{12} y^2 = 0 \end{aligned} \tag{89}$$

shows that there are in general four x-values for each value of y. There is no horizontal asymptote and the curve intersects the y-axis where

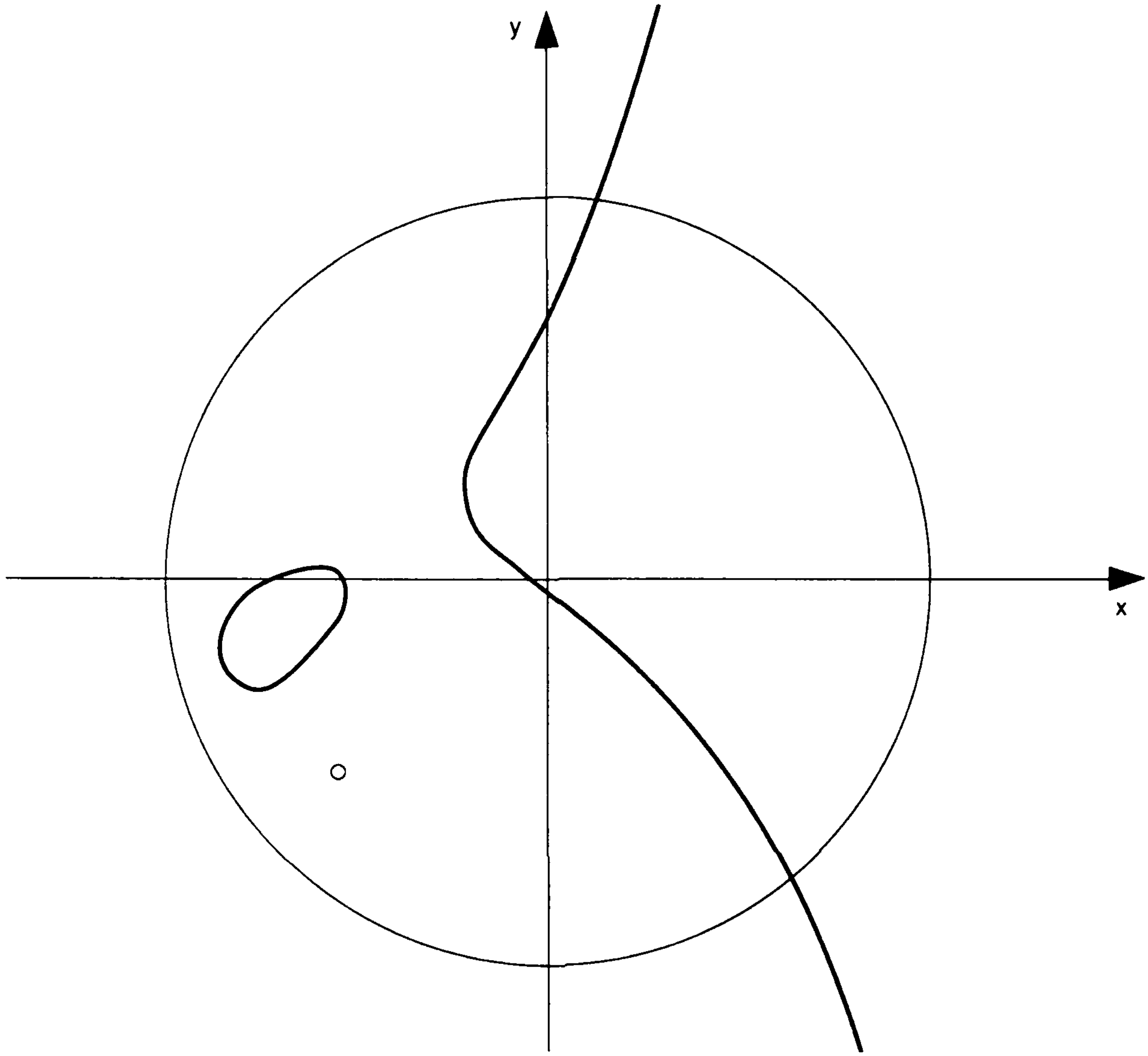
$$c_5y^2 + c_9y + c_{12} = 0, \tag{90}$$

that is, apart from the origin at two points at the most. Finally the t-equation

$$x^2\left[\left(c_5t^4 + c_4t^3 + c_3t^2 + c_2t + c_1\right)x^2 + \left(c_9t^3 + c_8t^2 + c_7t + c_6\right)x + c_{12}t^2 + c_{11}t + c_{10}\right] = 0 \tag{90}$$

indicates that there are in general only two curve points in each direction. Furthermore the origin is a double point and a possibility of four oblique asymptotes exists. Apart from these very general statements it is difficult to be specific about properties of the curve, except for the remark that the presence of four asymptotes seems incompatible with the fact that there are only two curve points in each direction.

To illustrate the locus in a specific case it is shown in Figure 11 together with Alhazen's curve. The locus intersects the circle at the same six points as does Alhazen's curve as should be expected. Since the points on the locus are subject to the same separation condition as the points of Alhazen's curve, we may apply the condition Eq. (85) to the locus also. The result is shown in Figure 12 where only four diffraction points remain. It is noted that the separation condition removes an entire branch of the locus and leaves a continuous curve defined solely by the coordinates of P_1 and P_2. In this case the curve indicates the radius, which separates the circles with two and four diffraction points. It is an interesting fact, that the locus may split into two separate curves, one of these being a closed curve. This is illustrated in Figure 13, where r = 10, $(x_l, y_l, z_l) = (5, 15, 10)$ and $(x_2, y_2, z_2) = (-5, -5, 4.75)$. The closed curve defines two circles with radii r_1 and r_2, where $r_1 < r_2$. Thus for the given location of P_1 and P_2 there will be two points of diffraction if the radius of the circle $r > r_2$ or if $r > r_1$. If $r_1 < r < r_2$ there will be four points of diffraction.



DIFFRACTION BY CIRCLE: INPUT DATA

x_1 =	5.000	y_1 =	15.000	z_1 =	10.000
x_2 =	−5.000	y_2 =	−5.000	z_2 =	4.750

DIFFRACTION POINTS AT FI = 82.81 311.28

COEFFICIENTS OF POLYNOMIALS

a =	−0.10000E+03	c =	−0.50000E+02	d =	−0.50000E+01	e =	0.00000E+00
f =	0.10000E+02	g =	−0.50000E+01	k =	0.50000E+02		
l =	−0.25766E+04	m =	−0.80700E+03	n =	0.19350E+04		
a_1 =	−0.20000E+04	a_2 =	−0.10000E+04	a_3 =	0.30000E+04	a_4 =	−0.10000E+04
a_5 =	−0.75766E+04	a_6 =	−0.66140E+04	a_7 =	0.69350E+04	a_8 =	−0.20000E+03
a_9 =	0.10000E+03	a_{10} =	0.00000E+00	a_{11} =	−0.50000E+03	a_{12} =	0.00000E+00
b_1 =	−0.20000E+04	b_2 =	−0.10000E+04	b_3 =	0.30000E+04		
b_4 =	−0.10000E+04	b_5 =	−0.27577E+05	b_6 =	0.33860E+04		
b_7 =	0.69350E+04	b_8 =	−0.50000E+05	b_9 =	0.00000E+00		

Figure 13. Diffraction by a circular edge. Separation condition applied. The curve shown is the locus.

5. CONCLUSION

In the present report the results of a preliminary investigation of some ray tracing problems related to circles and ellipses are described. The work is based on an alternative formulation of the law of reflection, that leads directly to an analytic solution to the problem of finding the points of reflection on the concave side of a spherical mirror. It is shown that the solution, referred to as Alhazen's solution, also applies to reflection by the convex side of a spherical mirror.

Furthermore a new concept in ray tracing, the locus, is introduced and applied to the circle problem. This leads to an alternative rule for the a priori determination of the number of reflection points and suggests a new numerical method for the determination of reflection points.

The techniques described above are applied to the ray tracing problems associated with reflection by an ellipse and diffraction by a circular edge. The solution of these problems is closely related to the geometry of plane algebraic curves given by polynomials in two variables. Table 1 shows the occurrence of the terms in the polynomials for the three cases considered, namely reflection by a circle and by an ellipse, and diffraction by a circular edge. It is seen that the governing equations become more complicated for the ellipse than for the circle, and that the diffraction problem is more involved than the reflection problems. This illustrates the fact that in the present cases, reflection may, from a ray tracing point of view, be considered a special case of diffraction.

For reflection by an ellipse, the locus method is modified and it is shown that no more than four reflection points are possible. In the diffraction case formally six solutions result, but two of these may be eliminated, since they correspond to non-physical configurations. Application of this observation may on the average halve the computer time spent on ray tracing in this case.

Further research is needed to assess the significance of the results found so far. Topics for further work include:

1. Simplifications resulting from the field point being at infinity.
2. Ray tracing for diffraction by an ellipse.
3. Multiple reflection and diffraction.
4. Application of an efficient numerical technique to the locus method.
5. Generalization to three dimensional problems.

Table 1. Occurrence of Terms in Algebraic Equations.

Term	x^4	x^3y	x^2y^2	xy^3	y^4	x^3	x^2y	xy^2	y^3	x^2	xy	y^2	x	y
Alhazen Circle Reflection										•	•	•	•	•
Locus Circle Reflection						•	•	•	•	•	•	•		
Alhazen Ellipse Reflection							•	•		•	•	•	•	•
Locus Ellipse Reflection		•		•		•	•	•	•	•	•	•		
Pseudo-Locus Ellipse Reflection						•	•	•	•	•	•	•		
Alhazen Circle Diffraction						•	•	•	•	•	•	•	•	•
Locus Circle Diffraction	•	•	•	•	•	•	•	•	•	•	•	•		

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